Robust Hybrid Output Regulation for Linear Systems with Periodic Jumps: Semi-classical Internal Model Design

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Abstract—A complete procedure for the design of a robust output feedback regulator is proposed for a class of uncertain linear hybrid systems with periodic jumps, using a hybrid extension of the classical internal model principle. Simple conditions, testable on the plant nominal data, for the problem solvability are given. The plant is not restricted to be minimum phase, square or single-input-single-output. The proposed regulator has a key feature of containing an internal model composed by two main units, a flow internal model, in charge of providing the correct input to achieve regulation during flows, and a jump internal model, in charge of suitably resetting the state of the regulator at each period. The proposed procedure is illustrated by its application to a physically motivated example, for which the output regulation problem is not solvable by methods appeared thus far in the literature.

I. INTRODUCTION

Control of hybrid systems, characterized by the interaction between a continuous-time (flow) dynamics and a discrete-time (jump) dynamics, has been widely studied in the last years, as shown, e.g., in [1], [2] and references therein. The problem of output regulation, well studied for linear [3], [4] and nonlinear [5], [6], [7], [8] systems, is also a very relevant problem in many applications involving hybrid systems. Due to several difficulties arising when trying to extend the classic output regulation theory to the hybrid setting, a specific scenario was identified in [9] in which hybrid phenomena arise without destroying the underlying linearity of the considered flow and jump dynamics; the study of the output regulation problem in the same setting was later addressed in several papers [10], [11], [12], [13], [14]. The assumption of [9] consists in considering that jumps only occur according to a periodic sequence of time instants, and that both the plant and the exosystem jump simultaneously. Despite its deceptively simple appearance, this problem formulation gives rise to a plethora of intriguing and unexpected phenomena that have no parallel in the classic regulation theory. The flow zero-dynamics internal model principle introduced in [13] is particularly relevant for the present discussion. The principle shows that, in general, the steady-state input achieving regulation contains a suitable copy of the modes of the flow zero dynamics of the plant. Such modes should be expected to be affected by uncertainties and then unknown. However, under some easily checkable and physically motivated structural conditions, it is possible to isolate a class of hybrid systems, namely semi-classical systems [15], [16], characterized by output-zeroing steady-state inputs that include only the natural modes of the exosystem. Such a class of systems is considered here. The robust version of the problem of hybrid output regulation proposed by [9] has been addressed in [11], [17]. While the contributions in [11], [17] focus on square, minimum-phase and (generally) relative degree one plants and Poisson stable exosystems, here no such assumptions are made, and the availability of more inputs than outputs is exploited to achieve solvability of the problem for a larger class of plants. The key difference in the two approaches consists in the allowed uncertainty affecting the plant, with the conditions in [11], [17] requiring a more structured uncertainty. The goal of this paper is to solve the problem of robust hybrid output regulation in the presence of additive uncertainties on the matrices of the state-space description of the plant. A preliminary version of the results was reported in [18]. The main novelty here is that sharper formal results are provided. Moreover, the compensator architecture has been revisited and simplified, thus yielding a clearer and more intuitive interpretation of its structure and inner working, as well as a simpler design procedure. Finally, the efficacy of the proposed compensator is illustrated by means of a physically motivated example, namely an RC electric circuit subject to periodic switchings whose physical parameters are allowed to vary arbitrarily in a neighborhood of their nominal values (the same system, but without parameter variations, was considered in [16]); it is worth stressing that the regulation problem for such a system cannot be solved by any other result currently available in the literature.

Notation. The acronym GES is used for Global Exponential Stability. OR for Output Regulation whereas LTI/LTV stand for Linear Time Invariant/Varying. $C_q := \{s \in \mathbb{C} : |s| < 1\}$. The Kronecker product is denoted by $\otimes$, and the spectrum of matrix $M$ by $\Lambda(M)$. For $r \in \mathbb{R}$, the shortcuts $\mathbb{N}_{\geq r} := \{m \in \mathbb{N} : m \geq r\}$ and $\mathbb{R}_{\geq r} := \{m \in \mathbb{R} : m \geq r\}$ are used. The degree of polynomial $p(s)$ is denoted $\deg(p(s))$. A pair of matrices $(A, B)$ is reachable if $[B, AB, \ldots, A^{n-1}B]$ is full row rank, and a pair of matrices $(A, C)$ is observable if $[C', A'C', \ldots, (A')^{n-1}C']$ is full column rank.

II. PRELIMINARIES AND PROBLEM DEFINITION

This paper focuses on a class of hybrid systems, introduced in [9], which experience periodic jumps separated by a flow
interval of known length $\tau_M > 0$, with the first jump occurring at some time $\varphi \in [0, \tau_M]$. A double time variable $(t, k)$ is used, where $t$ measures the flow of continuous-time and $k$ counts the number of jumps. Admissible values of $(t, k)$ belong to a hybrid time domain having the form:

$$T := \{(t, k) : t \in [t_k, t_{k+1}], k \in \mathbb{N}\},$$

$$t_k := \begin{cases} 0, & \text{if } h = 0, \\ \varphi + (h - 1)\tau_M, & \text{if } h \in \mathbb{N}_1. \end{cases}$$

For brevity, pre-jump and post-jump values of a signal will be denoted by a subscript as follows:

$$x(k) := x(t_{k+1}, k), \quad x(k) := x(t_k, k).$$

Global exponential stability (GES) for a system of the form

$$\dot{x}_a = A_a x_a, \quad x_a^+ = E_a x_a$$

with time domain $T$ means that there exist constants $c_1, c_2 \in \mathbb{R}_{>0}$ such that $\|x_a(t, k)\| < c_1 e^{-c_2 t} \|x_a(0, 0)\|, \forall t \in [0, \tau_M], \forall k \in \mathbb{N}$. Noting that $x_a(t, k) = e^{A_k \tau} E_k e^{A_k \tau} x_a(0, 0)$, for all $(t, k) \in T$ with $t \geq \varphi$, with $\bar{E}_a = E_a e^{A_k \tau}$ and $\tau = t - t_k$, it is easily seen that GES is equivalent to $\Lambda(\bar{E}_a) \subset C_1$. Hence, for this class of systems, stability can be assessed by a simple test on the eigenvalues of $\bar{E}_a$, without the need for a Lyapunov function; and, in particular, it is not necessary that $A_a$ be Hurwitz. Moreover, since eigenvalues are a continuous function of the matrix entries, GES in the nominal parameters implies GES in the perturbed parameters for small enough variations of $E_a$ and $A_a$. Consider the following hybrid linear plant $P$:

$$\dot{x} = Ax + Bu + Pw,$$

$$e = Cx + Qw,$$

$$x^+ = Ex + Rw,$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $e \in \mathbb{R}^p$, with $p \leq m$, represent the state, the input and the output of $P$; $w \in \mathbb{R}^q$ acts as an exogenous input and represents the state of an exosystem $E$:

$$\dot{w} = Sw,$$

$$w^+ = Jw.$$

Define the matrix $J = J e^{SrM}$. The following assumption rules out trivial cases, by guaranteeing that no signal generated by (5) for non-zero initial states asymptotically converges to zero.

**Assumption 1:** $\Lambda(J) \cap C_q = \emptyset$.

The goal of this paper is to design an output feedback, dynamic hybrid regulator

$$\dot{x}_c = A_c x_c + B_c e,$$

$$u = C_c x_c,$$

$$x_c^+ = E_c x_c,$$

with $x_c \in \mathbb{R}^{n_c}$, ensuring convergence to zero of the output $e$ for the interconnected system formed by (4), (5) and (6), as well as exponential stability of the interconnection of (4) and (6) with $w \equiv 0$, namely as in (3) with $x_a = [x', x_c']$ and

$$A_a = \begin{bmatrix} A & BC \\ B & C & A \end{bmatrix}, \quad E_a = \begin{bmatrix} E & 0 \\ 0 & E_c \end{bmatrix}.$$

Moreover, such goals must be achieved despite the presence of uncertainties in the plant description.

**A. The considered class of hybrid systems**

The plant $P$ in (4) is uncertain and belongs to a family $\mathcal{F}$ of admissible plants, specified in terms of a nominal plant $P^0$ and a family of admissible perturbations $\Delta \mathcal{F}$ to which each specific admissible perturbation $\Delta P$ belongs. The matrices in the state space description of $P^0$ are denoted as $(A^0, B^0, P^0, C^0, Q^0, E^0, R^0)$, whereas the perturbations affecting each matrix are denoted as $(\Delta A, \Delta B, \Delta F, \Delta C, \Delta Q, \Delta E, \Delta R)$; hence, the matrices in (4) are given by $A = A^0 + \Delta A$ and so on. The family $\Delta \mathcal{F}$ is characterized by specifying that, for all $P \in \mathcal{F}$, certain blocks must be zero both in the nominal and in the perturbed values of $A, B$ and $C$.

**Assumption 2:** $P^0 \in \mathcal{F}$, and for all $P \in \mathcal{F}$ the matrices in (4) have the partitioned form:

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} \\ B_{22} \\ B_{32} \end{bmatrix}, \quad P = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & C_3 \end{bmatrix}, \quad Q = Q, \quad R = \begin{bmatrix} R_{11} \\ R_{22} \\ R_{33} \end{bmatrix},$$

$$E = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix}.$$
robustly by means of a regulator containing only an internal model of the exosystem (that is, the flow zero dynamics internal model principle [13] is trivially satisfied in this class of plants).

B. Problem definition

Assume that the plant $P$ in (4) belongs to the family $F$ of plants and has a nominal description $P^0$.

**Problem 1:** Given the nominal plant $P^0 \in F$ as in (4) and (7), and the exosystem $E$ in (5), find, if possible, a robust output regulator as in (6) that achieves

- (GES) global exponential stability of the interconnected system formed by (4) and (6) with $u \equiv 0$;
- (OR) $\lim_{t+k \to +\infty} e(t, k) = 0$ for all initial states of the interconnected system formed by (4), (5) and (6);

for any $P \in \mathcal{F}_0 \subset F$, for some $\mathcal{F}_0$ containing an open neighborhood $B_r(P^0)$ of $P^0$ for some $r > 0$.

Note that in Problem 1 the regulator (6) must be designed using only the knowledge of the nominal plant $P^0$, but it must work for all plants in a set $\mathcal{F}_0 \subset F$ containing an open neighborhood of $P^0$. The stress on the open neighborhood $B_r(P^0)$ means that the regulator is able to achieve its objective for any perturbation of the non structurally zero elements in (7), provided that it is small enough that the perturbed plant remains in $B_r(P^0)$.

C. Structural conditions for output regulation

Consider e.g. the continuous-time regulation problem for plant (4a)-(4b) and exosystem (5a). Classic results show that the output regulation problem is solvable if and only if: $a_c$ the plant is stabilizable and detectable; and $b_c$ the Francis equation

$$II^S = AII + B\Gamma + P, \quad 0 = CII + Q,$$

is solvable. Condition $a_c$ is equivalent to the PBH tests:

$$\text{rank } \begin{bmatrix} sI - A & B \\ 0 & 0 \end{bmatrix} = n, \quad \forall s \in \text{Re}(s) \geq 0,$$

$$\text{rank } \begin{bmatrix} sI - A' & C' \end{bmatrix} = n, \quad \forall s \in \text{Re}(s) \geq 0,$$

whereas $b_c$ is implied by the non-resonance condition:

$$\text{rank } \begin{bmatrix} A - sI & B \\ C & 0 \end{bmatrix} = n + p, \quad \forall s \in \Lambda(S),$$

which becomes necessary under mild conditions on the uncertainties affecting the problem data.

In the hybrid case, the conditions for the solvability of Problem 1 are wholly analogous to $a_c$ and $b_c$, although the interaction between the flow and jump dynamics calls for more complex expressions [16], [13], [10]. The first condition is: $a_h$ the plant (4) is stabilizable and detectable. This condition is equivalent to satisfaction of the hybrid PBH tests ([12], [19]):

$$\text{rank } \begin{bmatrix} R_H(s) \\ O_H(s) \end{bmatrix} = n, \quad \forall s \notin C_g,$$

where

$$R_H(s) := [Ee^{A^{SM}} - sI \quad R_F],$$

$$O_H(s) := [(Ee^{A^{SM}})' - sI \quad O_F]'$$

and $R_F := [B \quad AB \quad A^2B \cdots A^{n-1}B]$ and $O_F := [C' \quad (CA)' \quad (CA^2)' \cdots (CA^{n-1})'].$

In these PBH tests, the interplay between flow and jump dynamics is evident in the presence of $R_F$, $O_F$ and $Ee^{A^{SM}}$. The second condition [16] is: $b_h$ the reduced flow Francis equation:

$$\Pi_3S = A_{33}\Pi_3 + B_{32}\Gamma_2 + P_3,$$

$$0 = C_3\Pi_3 + Q,$$

and the monodromy Francis equation:

$$\hat{\Pi}J = \hat{A}\hat{\Pi} + \hat{B}\hat{\Gamma} + \hat{P},$$

$$0 = \hat{C}\hat{\Pi} + \hat{D}\hat{Q},$$

must be solvable, where $\hat{A}, \hat{B}, \hat{C}, \hat{D}$ in (13) are given by:

$$\hat{A} := \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} \in \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad \hat{B} := \begin{bmatrix} E_{11} \\ E_{21} \end{bmatrix},$$

$$\hat{C} := \begin{bmatrix} E_{31} & E_{32} \end{bmatrix} \in \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}^T, \quad \hat{D} := E_{31},$$

whereas the expressions for $\hat{P}, \hat{Q}$ are found in [16]. Clearly, (12) is the classic Francis equation applied only to the subsystem of (4a), (4b) with state $x_3$ (which is the only part of (4) affecting the regulated output during flows); on the other hand, (13) is an invariance condition for the steady-state evolution arising as the hybrid complement to (12) on the equivalent discrete-time (“monodromy”) dynamics over one period [16], [10]. As in the non-hybrid case, (12) and (13) are solvable if the following non-resonance conditions hold:

$$\text{rank } (P_{F,3}(s)) = n_3 + p, \quad \forall s \in \Lambda(S),$$

$$\text{rank } (P_H(s)) = n, \quad \forall s \in \Lambda(J),$$

where

$$P_{F,3}(s) := \begin{bmatrix} A_{33} - sI & B_{32} \\ C_3 & 0 \end{bmatrix}, \quad P_H(s) := \begin{bmatrix} \hat{A} - sI & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix}.$$

Also in the hybrid case, (15) become necessary under mild conditions on the uncertainties affecting the problem data, e.g. under arbitrarily small but independent perturbations of the elements of a single column of $P_3$, $Q'$ or $R$.

**Assumption 3:** The following conditions:

$$\text{rank } (R_H(s)) = n, \quad \forall s \notin C_g,$$

$$\text{rank } (O_H(s)) = n, \quad \forall s \notin C_g,$$

$$\text{rank } (P_{F,3}(s)) = n_3 + p, \quad \forall s \in \Lambda(S),$$

$$\text{rank } (P_H(s)) = n, \quad \forall s \in \Lambda(J),$$

are satisfied by the nominal plant $P^0 \in F$.

Note that (17a), (17b) are, respectively, equivalent to $\text{rank } (R_H^0(s)) = n$ and $\text{rank } (O_H^0(s)) = n$ for all $s \in \Lambda(Ee^{A^{SM}})$, $s \notin C_g$.

III. REGULATOR ARCHITECTURE AND DESIGN

As shown in Fig. 1, the proposed regulator is composed of two main dynamical blocks: the “internal model component” $\hat{I}_M$ (in charge of ensuring a steady-state associated to zero regulated output), and a dynamic stabilizer $K$. The dynamic stabilizer $K$ is composed by
an essentially discrete-time\(^1\) controller \(\mathcal{K}_P\), designed to stabilize the system \(\Sigma\) in Fig. 1;  
• two interfacing dynamic components \(\mathcal{F}_e\) and \(\mathcal{F}_u\), in charge of making \(\Sigma\) stabilizable via an essentially discrete-time stabilizer.

The “internal model component” \(\mathcal{I}_M\) is composed by two subsystems \(\mathcal{I}_F\) and \(\mathcal{I}_J\), interconnected as in Fig. 2:  
• an essentially discrete-time jump internal model \(\mathcal{I}_J\), which is in charge of ensuring that the states of the subsystem \((A_{33}, B_{32}, C_3)\) and \(\mathcal{I}_F\) have suitable values after each jump;  
• a flow internal model \(\mathcal{I}_F\), whose state is reset at its input value at each jump time, which, suitably reinitialized at each jump time by \(\mathcal{I}_J\), is able to generate the input required for output regulation during each flow interval.

### A. The flow internal model \(\mathcal{I}_F\)

The flow internal model \(\mathcal{I}_F\) can be designed focusing exclusively on the flow dynamics of the exosystem and of the subsystem described by \((A_{33}, B_{32}, C_3)\). This subsystem has \(p\) outputs and \(m_2 \geq p\) inputs, and then it is not square when \(m_2 > p\). The \(p\) outputs impose that the internal model must contain \(p\) independent copies of the flow dynamics of the exosystem; then, a suitable matrix gain \(M_2 \in \mathbb{R}^{m_2 \times p}\) has to be designed when \(m_2 > p\) to generate an output \(y_F\) compatible in size with the plant input \(u_2\) and ensuring observability of the interconnected system.

**Algorithm 1: Design of \(\mathcal{I}_F\)(see Fig. 2)**

#### Step 1 (Design of the core flow internal model \(\mathcal{I}_{F0}\))

Let \(\mu_S(s)\) be the minimal polynomial of \(S\), and define \(n_S := \text{deg}(\mu_S(s))\). Let \(A_{F0} \in \mathbb{R}^{n_S \times n_S}\) be the lower companion matrix with \(\det(sI - A_{F0}) = \mu_S(s)\) and \(C_{F0} = [1 \ 0 \ \cdots \ 0] \in \mathbb{R}^{1 \times n_S}\). Define \(\mathcal{I}_{F0}\) according to

\[ \dot{x}_F = A_F x_F, \quad x^{\top}_F = u_{F0}, \quad y_{F0} = C_F x_F, \]  

where \(x_F \in \mathbb{R}^{n_F}\), \(n_F = pn_S\), \(A_F = I_p \otimes A_{F0}\), \(C_F = I_p \otimes C_{F0}\).

\(^1\)By essentially discrete-time signal we indicate a signal which is constant on each interval \([t_k, t_{k+1}].\) An essentially discrete-time system is a system unaffected by its input values during flows, and whose state and output signals are essentially discrete-time.

#### Step 2 (Design of the plant interconnection \(M_2\))

If \(m_2 = p\) then define \(M_2 = I_p\), otherwise find a matrix \(M_2 \in \mathbb{R}^{m_2 \times p}\), such that

\[ \text{rank}\left[ A_{33}^2 - sI \quad B_{32}^2 M_2 \right] = n_3 + p, \quad \forall s \in \Lambda(S) \quad (19) \]

#### Step 3 (Complete flow internal model \(\mathcal{I}_F\))

Define \(\mathcal{I}_F\) as the cascade interconnection of \(\mathcal{I}_{F0}\) and \(M_2\), i.e.

\[ y_F = M_2 y_{F0} = M_2 C_F x_F \]  

and let \(u_{F0} = y_J + u_F\) (see Fig. 2).

Note that the role of \(M_2\) consists in squaring down the flow dynamics described by \((A_{33}, B_{32}, C_3)\), so that the “squared” system \((A_{33}, B_{32} M_2, C_3)\) has exactly \(p\) inputs and \(p\) outputs and has no invariant zero in the set \(\Lambda(S)\), thus guaranteeing that the cascade interconnection of \(\mathcal{I}_{F0}\) (containing exactly \(p\) copies of the essential flow dynamics of the exosystem, as characterized by its minimal polynomial \(\mu_S(s)\)) and system \((A_{33}, B_{32} M_2, C_3)\) is observable. In turn, the flow observability property of the mentioned cascade of \(\mathcal{I}_{F0}\) and \((A_{33}, B_{32} M_2, C_3)\) (equivalently, of \(\mathcal{I}_F\) and \((A_{33}, B_{32}, C_3)\)) and classic output regulation theory imply the existence of a unique sequence of state values

\[ x_{3,[k]} = \Pi_3 w_{[k]}, \quad x_{F,[k]} = \Pi_F w_{[k]}, \quad (21) \]

where \(\Pi_3, \Pi_F\) solve a suitable Francis equation such that \(e(t, k) \equiv 0\) for all \(t \in [t_k, t_{k+1}]\).

Clearly, the exact values of \(\Pi_3\) and \(\Pi_F\) depend on the unknown plant parameters (and as such are used here as an analysis tool, but not needed in the regulator design). Hence, to have condition (21) satisfied for all \(k\), an ancillary robust output regulation problem must be solved, having a purely discrete-time nature and for a “regulated output”

\[ e_a(k) := \begin{bmatrix} x_{3,[k]} \\ x_{F,[k]} \end{bmatrix} - \begin{bmatrix} \Pi_3 \\ \Pi_F \end{bmatrix} w_{[k]}, \quad (22) \]

dimension \(n_3 + n_F\) (the number of scalar equations in (21)). These considerations give a qualitative motivation for the design of \(\mathcal{I}_J\) in the following subsection.

### B. The jump internal model \(\mathcal{I}_J\)

The jump internal model \(\mathcal{I}_J\) is designed focusing exclusively on the one period equivalent dynamics of the cascade of \(\mathcal{I}_F\) and \(\mathcal{P}\), and the one period equivalent dynamics of the exosystem, given by \(w_{[k+1]} = J w_{[k]}\). The internal model

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**Figure 1.** The internal model based regulator.

**Figure 2.** Structure of the internal model \(\mathcal{I}_M\).
in $\mathcal{I}_\mathcal{J}$ is designed to provide at each period the correct initialization to $x_{3,[k]}$ and $x_{F,[k]}$ to ensure $e(t,k) = 0$ for all $(t,k)$ with $t \in \{k, t_{k+1}\}$, see (21), (22). As discussed for the flow internal model $\mathcal{I}_\mathcal{F}$, the jump internal model $\mathcal{I}_\mathcal{J}$ must contain $n_3 + n_F$ independent copies of the monodromy dynamics of the exosystem, since (22) corresponds to $n_3 + n_F$ “regulated outputs”; on the other hand, $\mathcal{I}_\mathcal{J}$ has to generate $m_1 + n_F$ signals, that is $y_{1F} \in \mathbb{R}^{n_F}$ feeding $\mathcal{F}_t$ and $y_{J} \in \mathbb{R}^{m_1}$ which is summed to $y_{K,p_1}$ to form $u_1$. Then, a matrix gain $M_1(\cdot)$ has to be designed to output $y_J$ compatible in size with the plant input $u_1$ and ensuring observability of the interconnected system. In addition, a constant matrix $\bar{M}_1$ is obtained as a by-product of the algorithm, in Step 2 or in Step 3, and used in the following constructions, in particular Algorithm 3.

**Algorithm 2: Design of $\mathcal{I}_\mathcal{J}$ (see Fig. 2)**

**Step 1** (Design of the core jump internal model $\mathcal{I}_{\mathcal{J}0}$)

Let $\mu_j(s)$ be the minimal polynomial of $\bar{J}$, and define $n_J := \deg(\mu_j(s))$. Let $E_{j0} \in \mathbb{R}^{n_J \times n_J}$ be the lower companion matrix with $\det(sI - E_{j0}) = \mu_j(s)$ and $C_{j0} = [1 \ 0 \ \cdots \ 0] \in \mathbb{R}^{1 \times n_J}$. Define $\mathcal{I}_{\mathcal{J}0}$ according to

$$
\dot{x}_J = 0,
$$

$$
x_{J}^+ = E_{j}x_{J} + u_{J} = \begin{bmatrix} E_{J1} & 0 \\ 0 & E_{J2} \end{bmatrix} \begin{bmatrix} x_{J1} \\ x_{J2} \end{bmatrix} + \begin{bmatrix} u_{J1} \\ u_{J2} \end{bmatrix},
$$

$$
y_{J0} = C_{J}x_{J} = \begin{bmatrix} C_{J1} \\ C_{J2} \end{bmatrix} \begin{bmatrix} x_{J1} \\ x_{J2} \end{bmatrix},
$$

(23a)

(23b)

(23c)

where $x_{J} \in \mathbb{R}^{n_J}$, $n_J = (n_3 + n_F) \cdot n_J$, $E_{j} = I_{n_3 + n_F} \otimes E_{j0}$, $E_{J1} = I_{n_3} \otimes E_{J0}$, $C_{J1} = I_{n_3} \otimes C_{j0}$, $C_{J2} \in \mathbb{R}^{n_3 \times n_J}$ and $C_{J2} \in \mathbb{R}^{n_3 \times n_J}$.

**Step 2** (Design of a constant plant interconnection $M_1$)

Define $M_0 := \int_0^{\tau_M} e^{A_1(s)\tau}B_1^{s}d\tau \in \mathbb{R}^{n_{1,x} \times n_J}$. If

$$
\exists s \in \Lambda(\bar{J}) : \text{rank} \left[ \begin{array}{c} A^0 - sI \\ C^0 \end{array} \right] < n, \quad (24)
$$

then go to Step 3 (to design a time-varying $M_1(\cdot)$), otherwise find a matrix $M_1 \in \mathbb{R}^{m_1 \times n_{3,x}}$, such that

$$
\text{rank} \left[ \begin{array}{c} A^0 - sI \\ C^0 \end{array} \right] = n, \quad \forall s \in \Lambda(\bar{J}). \quad (25)
$$

Define $M_1 = M_0 M_1$ and go to Step 4.

**Step 3** (Design of a time-varying interconnection $M_1(\cdot)$)

Define $G_2^1 := \int_0^{\tau_M} e^{A_1(s)\tau}B_1^{s}d\tau \in \mathbb{R}^{n_{1,x} \times n_J}$. Find a matrix $M_1 \in \mathbb{R}^{m_1 \times n_{3,x}}$, such that

$$
\text{rank} \left[ \begin{array}{c} A^0 - sI \\ C^0 \end{array} \right] = n, \quad \forall s \in \Lambda(\bar{J}). \quad (26)
$$

Define $M_1(\tau) := B_1^{11} e^{A_1(\tau - \tau_M)\tau}M_1$.

**Step 4** (Complete jump internal model $\mathcal{I}_\mathcal{J}$)

Define $\mathcal{I}_\mathcal{J}$ as the cascade of $\mathcal{I}_{\mathcal{J}0}$ and $M_1$, i.e. let

$$
y_{J}(t,k) = M_1(t - k)y_{J0}(t,k), \quad (27)
$$

$M_1(t - k) \equiv M_1$ if a constant $M_1$ has been designed.

The role of (24) consists in detecting if a constant $M_1$ can be used or a time-varying $M_1(\cdot)$ is needed. If a constant $M_1$ can be used, its role is the same as $M_2$ for $\mathcal{I}_\mathcal{F}$, that is, a form of squaring down (see the discussion after Algorithm 1). On the other hand, if a time-varying $M_1(\cdot)$ is needed, it has the effect of “squaring up”, that is, the effect of virtually “enlarging” the number of inputs $u_1$ to the $x_1$ dynamics and, dually, the number of outputs $y_J$ of $\mathcal{I}_\mathcal{J}$ (see Fig. 2 and Fig. 1); this is evident whenever $m_1 < n_3$, which clearly implies that (24) is necessarily satisfied.

**C. The stabilizer $K$**

The design of an output feedback stabilizer for the interconnection as in Fig. 1 of $\mathcal{P}$ and $\mathcal{I}_\mathcal{M}$ (with the structure in Fig. 2, and designed by Algorithms 1 and 2), is now addressed by introducing suitable pre- and post-filters $\mathcal{F}_u$ and $\mathcal{F}_e$ which allow to design $K_\mathcal{D}$ as an essentially discrete-time system. The steps in the algorithm have the following goals:

- **Step 1** tests if the series of $\mathcal{I}_\mathcal{M}$ and $\mathcal{P}$ is stabilizable from the input $[u_J \ u_F]^T$ of $\mathcal{I}_\mathcal{M}$ alone, in which case $\mathcal{F}_u$ is not needed and the design proceeds with Step 3;
- **Step 2** Since $\mathcal{P}$ is stabilizable from $u$ (by (17a) in Assumption 3), $\mathcal{F}_u$ can be designed to ensure that the interconnection of $\mathcal{F}_u$, $\mathcal{I}_\mathcal{M}$, $\mathcal{P}$ is stabilizable from $[u_J^T \ u_F^T]^T$;
- **Step 3** is concerned with the design of the subsystem $\mathcal{F}_e$, which elaborates the regulated output $e$ to produce a signal $e_\xi$ such that $e_\xi(k)$ is equal to $e_\eta(k)$ in (22), that is the regulated output for the auxiliary problem mentioned around (22);
- **Step 4** aims at constructing a stabilizer based on a purely discrete time model of $\Sigma$ (see Fig. 1) from input $[u_J^T \ u_F^T]^T$ to output $e_\xi(k)$.

As for the last point, note from (18), (23), (30) and Figures 1 and 2 that the input of $\Sigma$ acts only at jump times, and then $x_{\Sigma,[k-1]}$ depends only on $u_{\Sigma(k)}$, and $y_{\Sigma(k)}$ depends only on $x_{\Sigma(k)}$, (where $x_{\Sigma} := [\bar{x}' \ y_\Sigma(t) \ y_\Sigma(t)]$, $u_{\Sigma} := [u_{\bar{x}}' \ u_J' \ u_F']$ and $y_{\Sigma} := e_\xi$). Hence $\mathcal{I}_\mathcal{M}$ and $\mathcal{F}_e$ act as a sort of generalized hold devices (which receive an external input at times $(t_k, t_{k+1})$ and “hold” and “modulate” such value through their flow dynamics at times $(t, k)$ with $t \in [t_k, t_{k+1}]$), whereas $\mathcal{F}_e$ behaves as a generalized sampler (associating to $e(t,k)$, with $t \in [t_k, t_{k+1}]$, the value of $e_\xi$ at the single time $(t_k + 1, k)$). In this view, $K_\mathcal{D}$ acts like a discrete-time controller in a classic sampled-data control system, whose role is to stabilize the monodromy equivalent of $\Sigma$, modeled as

$$
\bar{x}(k + 1) = \bar{E}_\Sigma \bar{x} + \bar{F}_\Sigma \bar{u}(k), \quad (28a)
$$

$$
\bar{y}(k) = \bar{C}_\Sigma \bar{x}(k), \quad (28b)
$$

with $\bar{x}(k) := x_{\Sigma,k}$, $\bar{u}(k) := u_{\Sigma(k)}$, $\bar{y}(k) := y_{\Sigma(k)}$, and matrices $(\bar{E}_\Sigma, \bar{F}_\Sigma, \bar{C}_\Sigma)$ given in (29).

**Algorithm 3: Design of $K$ (see Fig. 1)**

**Step 1** (Check if $\mathcal{F}_u$ is needed)

Define $M_0 := \int_0^{\tau_M} e^{A_0(s)\tau}B_0^{s}d\tau \in \mathbb{R}^{n_{1,x} \times n_3}$. If

$$
\exists s \not\in \mathbb{C}_g, \text{rank} \left[ \begin{array}{c} E^0 - sI \\ C^0 \end{array} \right] < n, \quad \forall s \not\in \mathbb{C}_g, \quad (29)
$$


then set $n_v = 0$ and go to Step 3 ($\mathcal{F}_u$ is not needed).

**Step 2** (Design of $\mathcal{F}_u$)

Define $G_F^o := \int_0^\tau e^{A^o \tau} B \circ B^o e^{A^o \tau} d\tau$. Find a matrix $M_F \in \mathbb{R}^{n_i \times n_v}$, such that

$$\begin{bmatrix} E^o e^{A^o \tau M} - s I & M^o \end{bmatrix} \begin{bmatrix} E^o G_F^o M_F \end{bmatrix} = n_v \forall s \notin \mathbb{C}_g.$$  

Define $K_f(\tau) := B^o e^{A^o (\tau M - \tau)} M_F$ and $\mathcal{F}_u$ as

$$\begin{cases} \hat{\nu} = 0, & \nu^+ = u_v, \\ y_{K,F}(t,k) = K_f(t - t_k)\nu(t,k), \end{cases}$$

where $y_{K,F} = [y'_{K,F1}, y'_{K,F2}]'$ (see Fig. 1).

**Step 3** (Design of $\mathcal{F}_e$)

Define $\mathcal{F}_e$ as

$$\begin{cases} \dot{\xi} = -A_{3F}^3 \xi + C_{3F}^3 e, & \xi^+ = 0, \\ e_{\xi} = G_{3F}^{-1} A_{3F}^3 \xi, \end{cases}$$

where $A_{3F} = \begin{bmatrix} A_{33} & B_{32} M_2 C_{F} \\ 0 & A_F \end{bmatrix}$, $C_{3F} = \begin{bmatrix} C_3 \end{bmatrix}$, and $G_{3F} = \int_0^\tau e^{A_{3F}^3 \tau} C_{3F}^3 e^{A_{3F}^3 \tau} d\tau$.

**Step 4** (Design of $\mathcal{K}_D$)

Let $(\mathcal{E}_S, \tilde{F}_S, \tilde{C}_S)$ in (28) be given by (29). Find $\tilde{K}_S$, $\tilde{L}_S$ such that $\Lambda(\mathcal{E}_S + \tilde{F}_S \tilde{K}_S) \subset \mathbb{C}_g$, $\Lambda(\mathcal{E}_S - \tilde{L}_S \tilde{C}_S \tilde{e}_S) \subset \mathbb{C}_g$, and define $E_D = \tilde{E}_S + \tilde{F}_S \tilde{K}_S - \tilde{L}_S \tilde{C}_S, F_D = \tilde{L}_S, C_D = \tilde{K}_S$. Define $\mathcal{K}_D$ as

$$\begin{cases} \dot{x}_D = 0, & x'_D = E_D x_D + F_D e_{\xi}, \\ y_K = C_D x_D, \end{cases}$$

with $y_K, \tilde{K}_S$ partitioned as $y_K = [y'_{K,F}, y'_{K,J}, y'_{K,J}]'$ and $\tilde{K}_S = [\tilde{K}_{S,F}, \tilde{K}_{S,J}, \tilde{K}_{S,J}]'$. Note that, thanks to the careful design of $\mathcal{L}_M$, $\mathcal{F}_u$ and $\mathcal{F}_e$, the pair $(\mathcal{E}_S, \tilde{F}_S)$ is stabilizable and the pair $(\mathcal{E}_S, \tilde{C}_S)$ is detectable, so that the construction of $\mathcal{K}_D$ at Step 4 can be completed by any alternative design, as far as $(E_D, F_D, C_D)$ describe a discrete-time system that stabilizes the (monodromy) discrete-time system given by $(\mathcal{E}_S, \tilde{F}_S, \tilde{C}_S)$.

**Lemma 1:** The hybrid system with periodic jumps:

$$\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ A_1 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \sigma,$$

is immersed in the hybrid system with periodic jumps:

$$\begin{bmatrix} \xi'_a \\ \xi'_1 \\ \xi'_2 \end{bmatrix} = \begin{bmatrix} 0 & B_a & 0 \\ 0 & E_1 & E_2 \\ 0 & 0 & E_3 \end{bmatrix} \begin{bmatrix} \xi_a \\ \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \sigma,$$

is in the hybrid system with periodic jumps:

$$\begin{bmatrix} \xi'_a \\ \xi'_1 \\ \xi'_2 \end{bmatrix} = \begin{bmatrix} 0 & B_a E_1 & B_a E_2 \\ 0 & E_1 & E_2 \\ 0 & 0 & E_3 \end{bmatrix} \begin{bmatrix} \xi_a \\ \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \sigma.$$
\[ v = \begin{bmatrix} C_0 & 0 & C_2 \end{bmatrix} \begin{pmatrix} \kappa_0 & \kappa_1 & \kappa_2 \end{pmatrix} \]  
(35c)

In particular, if \[ \begin{bmatrix} \kappa_1 & \kappa_2 \end{bmatrix} \begin{pmatrix} 0 \end{pmatrix} = \begin{bmatrix} \bar{\kappa}_1 & \bar{\kappa}_2 \end{bmatrix} \begin{pmatrix} 0 \end{pmatrix} \] then \[ \begin{bmatrix} \kappa_1 & \kappa_2 \end{bmatrix} \begin{pmatrix} v' \end{pmatrix} \begin{pmatrix} t, k \end{pmatrix} = \begin{bmatrix} \bar{\kappa}_1 & \bar{\kappa}_2 \end{bmatrix} \begin{pmatrix} v' \end{pmatrix} \begin{pmatrix} t, k \end{pmatrix} \] for all \( (t, k) \in T \), \( k \geq 1 \). Moreover, if also \( \kappa_{a,[0]} = B_2 \kappa_{1,[0]} \) then \[ \begin{bmatrix} \kappa_1 & \kappa_2 \end{bmatrix} \begin{pmatrix} v' \end{pmatrix} \begin{pmatrix} t, k \end{pmatrix} = \begin{bmatrix} \kappa_1 & \kappa_2 \end{bmatrix} \begin{pmatrix} v' \end{pmatrix} \begin{pmatrix} t, k \end{pmatrix} \] for all \( (t, k) \in T \).

To apply the lemma, note that since \( M_1(\tau) := B_1^0 e^{A_1^0(\tau_m - \tau)} M_1 \) and \( K_1(\tau) := B_2^0 e^{A_2^0(\tau_m - \tau)} M_2 \), it is possible to write \[ \begin{pmatrix} K_{f,1}(\tau) & M_1 C_{J,1} \end{pmatrix} = C_a e^{A_\tau} B_a \]
with \( \bar{A}_a = - \begin{bmatrix} A_0^0 & 0 \\ 0 & A_1^0 \end{bmatrix} \), \( \bar{B}_a = \begin{bmatrix} M_0^0 & 0 \\ 0 & M_1 C_{J,1} \end{bmatrix} \), \( \bar{C}_a = \begin{bmatrix} B_1^0 e^{A_1^0 \tau_m} & B_2^0 e^{A_2^0 \tau_m} \\ B_2^0 e^{A_2^0 \tau_m} & 0 \end{bmatrix} \) where \( B_0^0 = [B_1^0 \ B_2^0] \).

Then, considering the partition of (33) as in (34), the matrices describing the regulator (6) are given by

\[
A_c = \begin{bmatrix} \bar{A}_a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \bar{A}_{c,4} \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ 0 \\ \bar{B}_{c,2} \end{bmatrix},
\]
(36a)

\[
E_c = \begin{bmatrix} 0 & \bar{B}_a \bar{E}_{c,1} & \bar{B}_a \bar{E}_{c,2} \\ 0 & \bar{E}_{c,1} & \bar{E}_{c,2} \\ 0 & 0 & \bar{E}_{c,4} \end{bmatrix}, \quad C_c = \begin{bmatrix} \bar{C}_a & 0 & \bar{C}_{c,2} \end{bmatrix}.
\]
(36b)

Since the behaviour of the time-invariant regulator and of the time-varying regulator differ at most for \( (t, k) \) with \( t \in [0, t_1] \), \( k = 0 \), it is clear that asymptotic properties as (GES) and (OR) are unaffected.

**E. Main result: efficacy of the proposed solution**

The effectiveness of the above designed regulator is stated in the next theorem.

**Theorem 1:** Under Assumptions 1, 2 and 3, the algorithms in Sections III-A, III-B, III-C and III-D can be completed, yielding the regulator (6), (36) that solves Problem 1.

As in classical linear output regulation theory, the convergence to zero of the regulated output is generically ensured by the presence of the internal model as far as the closed-loop remains asymptotically stable (see e.g. [12]).

**Corollary 1:** Under Assumptions 1, 2 and 3, let the regulator (6) be designed as specified in Sections III-A, III-B, III-C and III-D. The above regulator guarantees (OR) in Problem 1 for all plants \( P \in \mathcal{F} \) satisfying (17c) in the perturbed parameters for which it achieves (GES).

In other words, given a regulator as in Theorem 1, and denoted by \( \mathcal{F}_s \subset \mathcal{F} \) the set of plants such that \( K \) provides GES for the closed-loop system, the same regulator solves Problem 1 for all the plants in \( \mathcal{F}_s \) that satisfy (17c) in the perturbed parameters (with the latter condition (17c) failing at most for a set of plants of measure zero in \( \mathcal{F} \)). Finally, note that for such plants arbitrarily large variations of matrices \( P, Q, R \) in (4) are allowed; in fact, such variations do not affect either GES or (17c).

**F. On structured uncertainties in output regulation**

In non-hybrid output regulation theory, there are two classic kinds of perturbations: i) unstructured (or independent) perturbations [3], where each element of the matrices describing the plant is independently perturbed; ii) structured (or coordinated) perturbations [21], where variations on different elements of the matrices describing the plant are correlated. The arising robust output regulation theories are well-known to be complementary, each having its own advantages and disadvantages (see [3], [21] for a detailed discussion). In the hybrid context, the key difference between the present contribution and those in [11, 21, 17, 22], relies in the class of uncertainties that are allowed to affect the plant according to the assumptions made in each paper; since such class is the same for each one of [11, 21, 17, 22], for brevity the comparison will be made directly only with [11]. The solution in [11] requires that, for all considered parameter variations, two conditions hold: first, a certain matrix \( R(\tau) \) can be expressed as \( R(\tau) = R'(\tau) \Upsilon(\mu) \) (see [11, eq. (25)]) where \( R'(\tau) \) only contains known functions of the time variable \( \tau \), whereas \( \Upsilon(\mu) \) is a parameter dependent constant matrix; second, the steady-state condition [11, eq. (14)-(15)] must be satisfied. Requiring such conditions to hold despite parameter variations implies suitably structured perturbations. In fact, as far as [11, eq. (25)] is concerned, \( R(\tau) \) depends on the modes of the flow zero dynamics of the plant (see the expressions of \( R(\tau) \) and \( \Omega(\tau) \) in [11, Appendix C]) so that variations must be assumed not to alter the eigenvalues of certain matrices (see e.g. the comment after [23, Proposition 1]). Moreover, considering [11, eq. (14)-(15)], it is noted that it is an overdetermined system of equations, so that even in the nominal parameter values its solvability is a non-generic situation; the fact that such non-generic property is satisfied under parameter variations imposes that the variations are coordinated.

Note that both conditions [11, eq. (14)-(15) and (25)] need to be checked for all considered parameter variations, and then are usually hard to check. In this paper, the easily checkable Assumption 2 is used to ensure that the condition [11, eq. (25)] is structurally satisfied (that is, even when perturbations are present, the modes of the flow zero dynamics of the plant are unobservable in \( R(\tau) \)); moreover, solvability of the overdetermined system of equations [11, eq. (14)-(15)] is replaced by solvability of the underdetermined system of equations (12)-(13), which is guaranteed to be solvable by Assumption 3 (which only involves the nominal parameter values and then can easily be tested). It is stressed that replacing [11, eq. (14)-(15)] with (12)-(13) is only possible by exploiting the fact that the plant has more inputs than outputs, and then it is not possible for the square plants of [11].

**IV. Example**

Consider the RC circuit described in [16], in which the voltage across a capacitor is required to track a hybrid reference. The hybrid nature of the problem is induced by the feature that three capacitors are interconnected periodically by means of switches. The nominal values of the parameters are taken from [16], and are assumed herein to be affected by independent variations; hence the solution proposed in [16], based on exact knowledge of the system, cannot be applied. We construct the internal model units as in Algorithm 1 and Algorithm 2, with \( n_F = 2 \), since \( p = 1 \), \( A_F = S \) and \( C_F = [1 \ 0] \), and \( n_J = 6 \).
where $E_{10}$ is a lower companion matrix with the last row equal to $[-1 - 1.6830]$, respectively. The rank conditions (19) and (25) hold with $M_2 = 1$ and $M_1 = 1$, while the monodromy matrix $E^T e^{A^T t_M}$ is asymptotically stable as a discrete-time system, hence $n_p = 0$, thus avoiding the use of the subsystem $F_u$. The subcompensator $K_D$ is designed as in Step 4 of Algorithm 3. The DLQR Matlab function is used to compute $K_S$ and $L_2$. For $K_S$, DLQR is applied to the pair $(1.5E_2, 1.5F_2)$ (so that the nominal closed-loop eigenvalues have modulus smaller than 2/3) and using $10I_4$ and $I_8$ as state and input weights, respectively, in the quadratic index to be minimized. Using duality, $L_2'$ is computed by DLQR applied to the pair $(2E_2', 2C_2')$ (so that the resulting eigenvalues of the estimation error dynamics have modulus smaller than 1/2) with both weights equal to the identity matrix. In Fig. 3, the results of three simulations, starting from null initial conditions for the plant and the compensator and $w(0, 0) = [1 0]^T$, are reported. The black lines are the graphs of the response of the nominal system, whereas the blue dashed lines show the response of the system with the following perturbed values of the parameters: $C_1 = 1.25$, $C_2 = 0.8$, $C_3 = 1.25$, $R_1 = 1.15$, $R_2 = 0.75$ and $R_3 = 1.85$; such variations range from 15% to 25% w.r.t. the nominal value. The gray dotted lines depict the time histories of the response of the plant with perturbed parameters in closed-loop with the purely feedforward input in [16], computed for the nominal parameters; as expected, such an input does not achieve regulation when parameters are perturbed. Finally, note that in the simulation the value of $\phi$ in (1) is $\phi = 0.35$, and the jumps are periodic with $\tau_M = 1$.

V. CONCLUSIONS

A solution for the robust output regulation problem has been given for a class of linear hybrid systems, with periodic jumps, for which the internal model design can be performed based on the exosystem dynamics only. In addition to regulation, the proposed design achieves global exponential stability of the closed-loop system and applies to MIMO, non-square and non-minimum phase systems. Interesting features of the proposed design include the presence of two internal models of the exosystem, one taking care of regulation during flows and the other solving a related regulation problem at jumps.

REFERENCES


Figure 3. Time histories of the plant state $x_3$ and of the tracking error. The desired reference for $x_3$ is shown in red solid line. Black solid lines denote the results of the simulation in the nominal plant parameters, blue dashed lines describe the case of internal models with perturbed plant parameters, gray dotted lines represent the feedforward solution from [16] in the presence of perturbed plant parameters.