Brief paper

On input allocation-based regulation for linear over-actuated systems

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A R T I C L E   I N F O

Article history:
Received 10 October 2013
Received in revised form 30 May 2014
Accepted 8 October 2014
Available online 10 January 2015

Keywords:
Output regulation
Input saturation
Input redundancy
Control allocation

A B S T R A C T

Results concerning the output regulation problem for over-actuated linear systems are presented in this paper. The focus is on the characterization of the solution of the full-information regulator problem for systems which are right-invertible (but not left-invertible) and the input operator is injective. The intrinsic redundancy in the plant model is exploited by parameterizing all solutions of the regulator equations and performing a static or dynamic optimization on the space of solutions. This approach effectively shapes the non-unique steady-state of the system so that the long-term behavior optimizes a given performance index. In particular, nonlinear cost functions that account for constraints on the inputs are considered, within the general form of a hybrid system assumed for the allocation mechanism. An example is given to illustrate the proposed methodology.

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1. Introduction

Traditionally, the presence of a redundant set of control inputs in a given control system (defined as the availability of a larger number of control inputs than regulated outputs) is addressed either by "squaring down" the plant model (Saberi & Sannuti, 1988) or by resorting to control allocation (Bodson, 2002; Harkegård & Glad, 2005; Johansen & Fossen, 2013). In particular, many variations on the theme of this latter methodology – quite popular in vehicular applications, noticeably flight control – assume that a virtual control input can be defined, which has the same dimension as the regulated output (see Johansen & Fossen, 2013 and references therein). The control strategy designed on the basis of this virtual input is then "distributed" across the redundant set of actuators via optimization of a given cost function. Notwithstanding the fact that multiple actuators are often necessary for technological reasons, an optimal design of the allocation stage has shown to lead to strong advantages in a broad range of applications (ranging from the aerospace to the automotive and several other industrial fields), both in terms of saturation handling (De Tommasi, Galeani, Pironti, Varano, & Zaccarian, 2011; Zaccarian, 2009) and in terms of fulfillment of more general performance goals (Boncagni et al., 2012; Cordiner, Galeani, Mecocci, Mulone, & Zaccarian, 2014; Passenbrunner, Sassano, & Zaccarian, 2012; Trégouët, Arzelier, Peaucelle, Pittet, & Zaccarian, in press; Zhou, Fiorentini, Canova, & Serrani, 2013).

For systems that are affine in the control, it is typically assumed that the input redundancy lies completely in the null-space of the input matrix. Clearly, this scenario does cover all the possibilities, as injective input operators can still be considered. Input redundancy with full-rank input operators has been termed weak input redundancy in Zaccarian (2009), where a taxonomy of over-actuated linear systems was proposed on the basis of the distinction between the null space of the control input matrix versus the null space of the multivariable DC gain of the plant model. In a state-space setting, weakly input redundant systems are characterized by multiple independently controllable state trajectories.
that are compatible with a given output. Specifically, the trajectories of the inverse model are not uniquely determined by the initial conditions, hence the possibility exists to modify redundant steady-state motions that are all compatible with a given output reference. This feature is exploited in this paper within the framework of (full-information) output regulation theory.

To the best of our knowledge, the output regulation problem for linear over-actuated systems has been investigated first in Sigthorsson and Serrani (2006) in the context of tracking control for a linearized model of a hypersonic aircraft, and later extended to encompass linear parameter-varying models within the considered application (Sigthorsson, Serrani, Bolender, & Domanski, 2009). The steady-state optimization for an input-redundant linear system with nonlinear output function has been considered in Johannsen and Sbarbaro (2005), with exosystem model restricted to pure integrators. For the same type of exosystem, the results in De Tommasi et al. (2011) and Zaccarian (2009) provide a framework allowing for nonlinear dynamic allocation solutions. This very framework has been in turn adopted in Serrani (2012), where the output regulation problem for strictly proper over-actuated LTI models is approached by resorting to a redundant servo-mechanism that directly allocates the trajectories of the plant inverse model. A different approach, aimed at achieving output regulation by exploiting nonlinear solutions of the linear Francis equations, has been considered in Galeani and Valmorbida (2013) for exosystems restricted to pure integrators, and in Valmorbida and Galeani (2013) for Poisson stable exosystems.

In this paper, we consider the full-information output regulation problem for non-strictly proper, over-actuated LTI models by restricting our attention to the weakly input-redundant case. The focus of the paper is on the characterization in geometric terms (and the ensuing parameterization) of the redundancies provided by an infinite number of solutions to the regulator equations. This parameterization is then exploited by an appropriate allocation mechanism, which in its most general form takes the structure of a hybrid system. The geometric properties of the solution of the regulator equations are then invoked to determine the most suitable structure of the compensator on the basis of the specific allocation policy adopted (i.e., constant or time-varying) and the corresponding behavior of the resulting reference motion.

The paper is organized as follows: background material is presented in Section 2, where the problem is formally stated. In Section 3 the properties of the solution of the regulator equation are discussed, and the structure of the allocation model is proposed. The selection of the allocation policy is discussed in Section 4. Finally, an illustrative simulation example is presented and discussed in Section 5, and conclusions are offered in Section 6.

**Notation:** For a matrix $S$, ker $S$ denotes its kernel, im $S$ denotes its image, and if $S$ is square, spec $(S)$ denotes its spectrum (the set of its eigenvalues). $C^0$ denotes the set of purely imaginary complex numbers.

**2. Background and problem statement**

We consider linear systems of the form

$$\dot{w} = Sw \quad (1a)$$

$$\dot{x} = Ax + Bu + Pw$$

$$e = Cx + Du + Qw \quad (1c)$$

with state $w \in \mathbb{R}^q$ and $x \in \mathbb{R}^m$, control input $u \in \mathbb{R}^p$ and performance output $e \in \mathbb{R}^p$. Following standard terminology in output regulation theory (Davidson & Goldenberg, 1975; Francis, 1977), $P := (A, B, C, D)$ is referred to as the realization of the plant and $\delta := (S, P, Q)$ as the realization of the exosystem. The following assumptions define the class of models considered in this paper.

**Assumption 1.** (1) $P$ is over-actuated, $m > p$;
(2) rank $B = m$ and rank $C = p$;
(3) $(A, B)$ is stabilizable;
(4) The matrix $S$ is semi-simple (that is, it has only simple eigenvalues) and spec $(S) \subset C^0$.

**Remark 1.** Item 2 of Assumption 1 is required to avoid trivialities and overlap with previous results in Zaccarian (2009). The case of a rank-deficient $B$, which corresponds to having the so-called strong input redundancy for $(A, B)$, can be handled separately from the weak input redundancy exploited here, essentially by projection modulo ker $B$.

The problem addressed in this paper is the design of a full information (possibly nonlinear) regulator that is capable of exploiting the redundancy stated at item 1 of Assumption 1 to induce a desirable selection of the control input $u$ (in a sense to be specified). As customary, by full information it is meant that both $x$ and $w$ are available for measurement. Here, it is also assumed that $P$ and $\delta$ are known exactly. As for the possibilities offered by over-actuation, we consider in Section 4 the minimization of a functional that corresponds to keeping the steady state input away from the saturation limits. As pointed out in Zaccarian (2009), the use of input allocation in the presence of input saturation should be seen as synergistic with anti-windup techniques, since the latter account for saturation during transients, whereas the former addresses steady-state saturations. This must be accomplished while guaranteeing asymptotic stability of the controllable modes of $(1)$ and the asymptotic tracking requirement $\lim_{t \to \infty} e(t) = 0$.

Define the system matrix of $P$ as $P_S(s) := \begin{bmatrix} A - st & B \\ C & D \end{bmatrix}$ (see Rosenbrock, 1970). Under Assumption 1, a well-known sufficient condition for the solvability of the regulator problem, which is necessary if generic solvability is considered (i.e., for all matrices $P \in \mathbb{R}^{n \times m}$ and $Q \in \mathbb{R}^{p \times q}$), is given by:

**Assumption 2.** The set of invariant zeros of $P$ is disjoint from the spectrum of $S$, that is, rank $P_S(\lambda) = n + p$, $\forall \lambda \in$ spec $(S)$.

Assumption 2 implies that $P$ is non-degenerate (Hewer & Martin, 1984); this, together with Assumption 1.1, implies that $P$ is right-invertible. Recall that $P$ is left (right) invertible if and only if rank $P_L(s) = n + m$ (rank $P_R(s) = n + p$) as a polynomial matrix; obviously, an over-actuated system is not left-invertible. Left invertibility is equivalent to the fact that the applied input can be uniquely recovered from the forced response output, whereas right invertibility implies that any sufficiently smooth function can be reproduced as a forced output response.

Finally, we recall a few geometric concepts that will be used in the sequel. By $V^* \subset \mathbb{R}^n$, we denote the weakly unobservable subspace for $P$, i.e., the set of initial conditions for which there exists an input function such that the ensuing output is identically zero. It is well known (Trentelman, Stoorvogel, & Hautus, 2001) that $V^*$ is the largest subspace $V \subset \mathbb{R}^n$ such that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} V \subset (V \times 0) + \text{im} \begin{bmatrix} B \\ D \end{bmatrix},$$

or equivalently the largest subspace $V \subset \mathbb{R}^n$ such that there exists $F \in \mathbb{R}^{m \times n}$ ensuring

$$(A + BF)V \subset V, \quad (C + DF)V = 0. \quad (3)$$

A matrix $F$ satisfying $(3)$ is called a friend of $V$. Similarly, we denote by $R^* \subset \mathbb{R}^n$ the controllable weakly unobservable subspace\(^1\) of $P$,

\(^1\) When $D = 0$, $V^*$ and $R^*$ are usually termed respectively the largest controlled-invariant subspace and the largest controllability subspace contained in ker $C$ (Wonham, 1985).
i.e., the set of initial conditions for which there exists an input function able to steer the state to zero in finite time while keeping the output identically zero. Obviously, $\mathbb{R}^n \subset \mathcal{V}^*$; moreover, any friend of $\mathcal{V}^*$ is also a friend of $\mathbb{R}^n$ (Trentelman et al., 2001, Th. 7.14).

It is worth recalling that the set of invariant zeros of $\mathcal{P}$ coincides with the spectrum of the map induced in $\mathcal{V}^*/\mathbb{R}^n$ by $\mathcal{A}_p := A + BF$, where $F$ is a generic friend of $\mathcal{V}^*$.

3. Regulator architecture and zero error, steady-state solutions

It is well known (see, for instance, Davison & Goldenberg, 1975) that the structure of a full-information regulator comprises two independently designed compensators generating the control input

$$ u = u_a(w) + \bar{u}(\tilde{x}) $$

where

(i) $u_a : w \mapsto u_a(w) \in \mathbb{R}^m$ is a steady-state control action inducing an identically zero output $e$ along a suitable steady-state motion of the plant, $x_a : w \mapsto x_a(w) \in \mathbb{R}^n$;

(ii) $\bar{u} : \mathbb{R}^n \to \mathbb{R}^n$ is a stabilizing feedback from the mismatch $\tilde{x} = x - x_a(w)$, vanishing at $\bar{x} = 0$ and capable of stabilizing the steady-state motion $(x_a(w(t)), t \geq 0)$.

Since the pair $(x_a, u_a)$ describes a steady-state solution, it must satisfy (1b) almost everywhere along the trajectories of exosystem (1a), with $x_a(w(\cdot))$ an absolutely continuous function; hence the dynamics of $\tilde{x}$ are simply given by $\dot{\tilde{x}} = \Lambda \tilde{x} + \bar{B} \bar{u}$, so that a natural choice in item (ii) is the linear feedback

$$ \ddot{u} = \bar{F} \tilde{x} = F(x - x_a(w)), $$

where $F \in \mathbb{R}^{m \times n}$ is a stabilizing gain for the pair $(A, B)$, perhaps enjoying additional properties to be specified later. One of the contributions in this paper consists in showing how the same dynamic of $\tilde{x}$ can be followed even when an infinite family of steady state pairs $(x_a, u_a)$ is available so that an optimal solution can be pursued.

3.1. The solution set of the regulator equations

In the over-actuated case, since the plant model fails to have a unique inverse, redundancy can be exploited in the generation of the steady-state pair $(x_a, u_a)$. For linear models, this corresponds to selecting appropriately

$$ x_a(w) = \Pi w, \quad u_a(w) = \Gamma w $$

among the infinitely many solutions $(\Pi, \Gamma)$ to the regulator equations (Francis, 1977):

$$ \Pi S = A\Pi + B\Gamma + P $$

$$ 0 = C\Pi + D\Gamma + Q. $$

According to the proposition below, whose proof is given in Appendix B, all steady-state pairs in (6) can be generated by exploiting a basis of the space of all solutions to the homogeneous regulator equations:

$$ \Pi S = A\Pi + B\Gamma $$

$$ 0 = C\Pi + D\Gamma. $$

Proposition 1. Under Assumption 1, all solutions to the regulator equations (7) are parameterized as

$$ \Pi(\theta) = \Pi_p + \sum_{i=1}^s \theta_i \Pi_i, \quad \Gamma(\theta) = \Gamma_p + \sum_{i=1}^s \theta_i \Gamma_i, $$

by the parameter vector $\theta = [\theta_1 \cdots \theta_s]^T \in \mathbb{R}^s$, where $s = (m - p)q, x_s = [\Pi_s; \Gamma_s]$ is any solution$^2$ to (7), and $X_i = [\Pi_i; \Gamma_i], i = 1, \ldots, s$ are linearly independent matrices spanning the space of solutions to (8).

The next result, which is also proven in Appendix B, suggests additional useful properties of the solutions of (8).

Proposition 2. Each solution $X = [\Pi; \Gamma]$ of (8) satisfies $\Pi \in \mathbb{R}^n$.

3.2. Steady-state solution and allocation

The results of Propositions 1 and 2, as well as the general structure of a (static) full-information regulator mentioned above, suggest the architecture of the regulator with dynamic allocation shown in Fig. 1. In particular, dynamic allocation of the control input $u$ is performed by acting on the allocation parameter $\theta$. In order to consider a rather general class of functions for the allocation parameter $\theta$ (but reasonable for implementation), we define a sequence of times $[t_1, \ldots, t_N]$ possibly with $N = +\infty$ such that (denoting $\theta(0) = 0$, $\theta(\cdot)$ is differentiable in $(t_{j-1}, t_j)$ and is discontinuous at the jump times $t_j$, for all $j = 1, \ldots, N$. A quite convenient tool for enforcing this selection and the ensuing closed loop solutions is the hybrid dynamical systems framework developed in Goebel, Sanfelice, and Teel (2012). Such an approach leads to a trajectory for $\theta = \theta(t, k)$ defined over the hybrid time domain $T := \{t, k\} \in \mathbb{R} \times \mathbb{N} : t \geq 0, k = 0, 1, \ldots, N\), which is a solution to a hybrid system of the form:

$$ \dot{\theta} = \vartheta_{\tau}(\theta, w), $$

$$ \dot{\tau} = \vartheta_{\tau^+}(\theta, w), $$

$$ \tau = \tau^-, $$

depending on the timer $\tau$ and on the logic variable $q$, where the vector fields $\vartheta_{\tau}(\cdot)$ and $\vartheta_{\tau^+}(\cdot)$ are the available degrees of freedom. When using the hybrid dynamics (10) to generate $\theta(\cdot)$ and, consequently, (9), it will be convenient either to adopt the following simple expression for (6)

$$ x_a(\theta, w) = \Pi(\theta) w, \quad u_a(\theta, w) = \Gamma(\theta) w $$

(11)

or, equivalently, the tensor notation

$$ x_a(\theta, w) = \left[ \begin{array}{c} \Pi_1 \Pi_2 \cdots \Pi_s \end{array} \right] (\theta \otimes w) $$

(12a)

$$ u_a(\theta, w) = \left[ \begin{array}{c} \Gamma_1 \Gamma_2 \cdots \Gamma_s \end{array} \right] (\theta \otimes w) $$

(12b)

to emphasize bi-linearity in $\theta$ and $w$.

In order for the ensuing trajectory $x_a(\theta(\cdot), w(\cdot))$, in (11) to be a (steady-state) solution to (1b), the functions $\vartheta_{\tau}$ and $\vartheta_{\tau^+}$ must satisfy two compliance conditions. Firstly, $x_a(\theta(\cdot), w(\cdot))$ must be continuous across jumps: Using the jump equation in (10), and noting that $w(\cdot)$ does not jump, one needs to ensure that

$$ x_a(\theta) = \Pi(\theta) w + \Pi(\theta) w^+ - \Pi(\theta) w^- $$

$$ = \sum_{i=1}^s \vartheta_{\tau}(q, w) \Pi_i w = 0. $$

(13)

Secondly, (1b) must be satisfied during flows with $u = u_a(\theta, w)$, that is, according to (7) and (11),

$$ \dot{x}_a = \Pi(\theta) w + \dot{\Pi}(\theta) w = \Pi(\theta) w^+ + \dot{\Pi}(\theta) w $$

$$ = A\Pi(\theta) w + B\Gamma(\theta) w^+ + Pw + \dot{\Pi}(\theta) w $$

$$ = A\Pi(\theta) w + B\Gamma(\theta) w^+ + Pw, $$

$^2$ We use the notation $z = [x; y]$ to denote the vector (or matrix) $z = [x^T \ y^T]^T$. 

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which implies \[ \dot{\Pi}(\theta) w = \sum_{i=1}^{s} \dot{\theta}_i(\tau, w) \Pi_i w = 0. \] Summarizing, this last constraint and constraint (13) yield

\[
\begin{bmatrix}
\Pi_1 & \Pi_2 & \cdots & \Pi_s
\end{bmatrix} \begin{bmatrix}
\dot{\theta}_1(q, w) \\
\dot{\theta}_2(q, w) \\
\vdots \\
\dot{\theta}_s(q, w)
\end{bmatrix}
\begin{bmatrix}
\Pi_1 w \\
\Pi_2 w \\
\vdots \\
\Pi_s w
\end{bmatrix} = 0 \tag{14a}
\]
\[
\begin{bmatrix}
\Pi_1 & \Pi_2 & \cdots & \Pi_s
\end{bmatrix} \begin{bmatrix}
\dot{\theta}_1(r, w) \\
\dot{\theta}_2(r, w) \\
\vdots \\
\dot{\theta}_s(r, w)
\end{bmatrix}
\begin{bmatrix}
\Pi_1 w \\
\Pi_2 w \\
\vdots \\
\Pi_s w
\end{bmatrix} = 0. \tag{14b}
\]

Note that (14a) is trivially satisfied if \( \dot{\theta}() \) is continuous across jumps (that is, when \( \dot{\theta}_i \equiv 0 \)), and (14b) is trivially satisfied if \( \dot{\theta}() \) is constant during flows (that is, when \( \dot{\theta}_i \equiv 0 \)). In particular, such conditions are satisfied in the classical case where a constant value for \( \theta \) is used.

3.3. A regulator with allocation

Even if the use of a steady-state solution is usually more appealing, it is worth mentioning that nice regulation properties can be achieved even if the constraints (14) are not satisfied. In such a case, the dynamics of the mismatch \( \delta x \) between \( x \) and the quasi steady-state solution \( x_{ss}(\theta(), w()) \) that results from a hybrid generator (10) whose vector fields do not satisfy either one of conditions (14) (or both) is given by

\[
\dot{\delta} x = A \delta x + B \delta u - \sum_{i=1}^{s} \dot{\theta}_i \Pi_i w
\]
\[
\dot{\delta} x = \dot{\delta} x - \sum_{i=1}^{s} (\dot{\theta}_i - \dot{\theta}_i) \Pi_i w \tag{15}
\]
\[
e = C \delta x + D \delta u.
\]

Taking into account Proposition 2 and choosing \( F \) as a friend of \( \mathcal{R}^s \), it can be shown that the terms depending on \( \theta \) and \( \dot{\theta}() = \dot{\theta}(\tau, \theta) \) in (15) have no effect on the output \( e \) (although they still prevent the solution \( x() \) from converging to a "steady-state" motion). Based on the previous discussion, \( F \) shall be selected as follows:

(1) as any stabilizing gain for the pair \((A, B)\), under the condition that (14) is satisfied, e.g., when the allocation parameter \( \theta \) is statically optimized or is kept constant;

(2) as a stabilizing gain for \((A, B)\) which is also a friend of \( \mathcal{R}^s \); this second choice allows for arbitrary time-varying selections of the allocation parameter \( \dot{\theta}(\cdot) \) without affecting the regulation performance.

The two possible selections above and the desirable properties of the ensuing “Over-Actuated Full Information Regulator” in Fig. 1 are formally stated in Theorem 1, whose proof is given in Appendix B.

**Theorem 1.** Consider the system (1), satisfying Assumptions 1 and 2, and the controller selection in (4)–(6), where \((\Pi_1, \Gamma_1)\) and \((\Pi_1, \Gamma_1)\) for all \( i = 1, \ldots, s \) are computed as in Proposition 1, and the matrix \( F \) in (5) is such that \( A + BF \) is Hurwitz. Let \( \dot{\theta}(\cdot) \) be a locally essentially bounded solution to (10), and consider the following two cases:

(1) The hybrid generator (10) satisfies (14); or,

(2) the matrix \( F \) is also a friend of \( \mathcal{R}^s \).

For each one of the above cases, given any initial condition \((w(0), x(0))\) of (1), the trajectories of the closed-loop system are essentially bounded and the error satisfies \( \lim_{t \to \infty} e(t) = 0 \). Furthermore, for any initial conditions satisfying \( x(0, o) = \Pi(\theta(0, o), w(0, o)) \), all solutions satisfy \( e(t, j) = 0 \) for all \( (t, j) \in \text{dom}(e) \).

**Remark 1.** Although this paper focuses on the full information setting, the output feedback case can be easily dealt with under standard detectability hypotheses for the cascade of \( \mathcal{S} \) and \( \mathcal{P} \), by using an asymptotic observer. Similarly, by judicious choice of the performance index, the proposed optimization-based allocation policy (detailed in the next section) allows the designer to consider several alternative scenarios, e.g., to assign a higher priority to an input channel yielding a lower relative degree. An iterative process for the selection of the performance index (analogous to the one employed in linear quadratic regulation) can then be devised in order to accommodate competing requirements.

4. Selection of the allocation policy

Theorem 1 yields two possible design frameworks for \( \theta \), providing output regulation under suitable hypotheses. In each of the two frameworks, infinitely many steady-state input and state responses are available, so that there is an opportunity to optimize a suitable performance index. The purpose of this section is to exploit the framework from Theorem 1 in order to provide an optimization approach geared toward minimizing the \( L_\infty \) norm of the steady-state input \( u_{ss} \), under the simplifying assumption that \( w \) is periodic.

**Assumption 3.** The exosystem (1a) generates periodic responses, that is, there exists \( T > 0 \) such that for any \( w(0) \) and for all \( t \geq 0 \), \( w(t + T) = w(t) \).

Under Assumption 3, it is both reasonable and desirable to require \( \dot{\theta}(\cdot) \) as well as the steady-state response of (1) to be periodic. Since \( \dot{\theta}(\cdot) \) is defined on a hybrid time domain \( \mathcal{T} \), periodicity means that there exists a positive integer \( K \) such that \( \mathcal{T} = (T, K) \)-periodic (namely, \((t, k) \in \mathcal{T} \) implies \((t + T, k + K) \in \mathcal{T} \)) and \( \dot{\theta}(\cdot) = (T, K) \)-periodic (namely, \((t, k) \in \mathcal{T} \) implies \((t + T, k + K) = \dot{\theta}(\cdot)) \). For compactness in later expressions, define \( \mathcal{K} := \{0, 1, \ldots, K - 1\} \) and \( \mathcal{K}_{-} := \{1, \ldots, K - 1\} \). In view of periodicity, in order to optimize the steady-state response of interest it is enough to focus on one period, together \((T, k)\) ranging in the compact set \( \mathcal{T}_0 := \mathcal{T} \cap (\{0, T\} \times \mathcal{K}) \). Letting \( t_j \in \mathcal{K}, j \in \mathcal{K}, \) indicate the jump times, \( \mathcal{T}_0 \) can be explicitly described as \( \mathcal{T}_0 := \bigcup_{j=1}^{K-1} \{[t_j, t_{j+1}) \times \{h\} \} \) with \( t_0 = 0 \) and \( t_K = T \). Given \( w_0 \), the proposed optimization approach will thus compute functions \( \tilde{\theta}_j(t, w_0) \) and \( \tilde{\theta}_k(k, w_0) \) defined for \((t, k) \in \mathcal{T}_0 \) and the generation of \( \dot{\theta}(\cdot) \) by the hybrid system structure (10) will be achieved by setting

\[
\begin{align}
\dot{\theta}_j(t, w) &= \tilde{\theta}_j(t, e^{-ST}w), \quad T = (mod T), \\
\dot{\theta}_k(q, w) &= \tilde{\theta}_k(q, e^{-ST}w), \quad q = (mod K),
\end{align}
\]

where periodicity implies that \( w_0 = e^{-ST}w = e^{-ST}w \). Apart from the need to comply with the feedback form of \( \dot{\theta}_j \) and \( \dot{\theta}_k \) in (10), the term \( e^{-ST}w = w_0 \) in (16) is constant and need not be recomputed each time in application.

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3 A situation suggested by an anonymous reviewer, quite typical in applications.

4 Due to periodicity, one may reformulate (10), (16) in such a way that \( q, r \) evolves in the compact set \([0, T] \times \mathcal{K}.\)
4.1. An infinite dimensional optimization problem

Let \( \theta_T \) be the set of possible selections of \( \theta(\cdot) \) defined on a finite grid (to be chosen later) \((T, K)\)-periodic time domain \( T \). For the selection of an optimal \( \theta \in \theta_T \), we consider the performance index (note that the motion of \( w(\cdot) \) is completely determined by the initial condition \( w(0) = w_0 \))

\[
J(\theta, \omega_0) = \| u_\omega(w, \theta) \|_{\infty, T},
\]

where \( \| v \|_{\infty, T} = \max_{t \in [0, T]} |v(t, k)|_\infty \) and \( |v|_\infty := \max_{n \in \{1, \ldots, n\}} |v|_n \) for \( v \in \mathbb{R}^n \). If a “true” steady-state solution is desired, it is necessary to consider a constraint analogous to (14) but formulated directly on the signal \( \theta(\cdot) \), namely for \( k \in \mathbb{K} \), and almost all \( t \) such that \( (t, k) \in T_0 \),

\[
\left( \Pi_1 \Pi_2 \cdots \Pi_t \right) \left( \delta_k \otimes w(t_k, k) \right) = 0,
\]

where \( \delta_k = \begin{cases} \theta(0, 0) - \theta(T, k), & k = 0, \\ \theta(t, k) - \theta(t, k-1), & k \in \mathbb{K} \end{cases} \).

The goal of this section can be formalized as follows:

**Problem 1.** Find, if possible, \( \theta^* : (t, k) \rightarrow \mathbb{R}^3 \) such that

\[
J(\theta^*, \omega_0) = \left( \min_{\theta \in \theta_T} J(\theta, \omega_0) \right) s.t. (18).
\]

In order to suitably reformulate Problem 1, given a scalar \( \alpha \in \mathbb{R} \), define the constraints

\[
-\alpha \leq u_{\omega}(w(t, k), \theta(t, k)) \leq +\alpha, \quad \forall (t, k) \in T_0,
\]

and consider the following alternative problem.

**Problem 2.** Find, if possible, \( \theta^* : (t, k) \rightarrow \mathbb{R}^3 \) such that

\[
J(\theta^*, \omega_0) = \left( \min_{\theta \in \mathbb{E}_T, \alpha} \alpha \right) s.t. (18), (19).
\]

The following fact is easily seen to hold due to the relation \( J(\theta, \omega_0) = \min \alpha \) s.t. (19).

**Fact 2.** Problem 1 is equivalent to Problem 2.

4.2. A finite dimensional relaxation

The formulation in Problem 2 clearly shows that the minimization problem of interest is infinite dimensional (although linear), both due to the infinite dimensionality of the space of solutions for \( \theta(\cdot) \) and to the infinitely many constraints (18b), (19). Motivated by the necessity to simplify the analysis and obtain a practical implementation, a relaxed problem will be considered. The relaxation consists first in restricting the set \( \theta_T \) of admissible selections of \( \theta(\cdot) \) to a class of easily implementable functions (which makes (18b) trivially satisfied), and then in imposing (19) only at a finite set of times. Specifically, given a \((T, K)\)-periodic time domain \( T \), the class \( \theta_T \) will be chosen as the set of all \((T, K)\)-periodic, piecewise constant functions \( \theta \) defined on \( T \), finely parameterized by \( \delta_k, k \in \mathbb{K} \), as

\[
\theta(t, k) = \delta_k + \begin{cases} \theta_{t_k}, & t \in [t_k, t_{k-1}], \ k \in \mathbb{K}, \\
\end{cases}
\]

where \( t_k = T, \) so that the \( \delta_k \)'s in (18a) become \( \delta_0 = \theta_0 - \theta_{k-1} \) and \( \delta_k = \theta_{k} - \theta_{k-1} \) for \( k \in \mathbb{K} \). With this choice of \( \theta_T \), constraint (18b) is trivially satisfied (hence it can be omitted).}

Regarding constraint (19), we may approximate it by way of gridding of \( T_0 \) by selecting an integer \( H \geq K \) and introducing \( \mathcal{H} := \{0, 1, \ldots, H-1\} \) and

\[
\delta_0 := \{(s_h, \omega_h), \ h \in \mathcal{H} \},
\]

and relax (19) by replacing the infinite set \( T_0 \) by the finite set \( \delta_0 \):

\[
-\alpha \leq u_{\omega}(w(s, \sigma), \theta(s, \sigma)) \leq +\alpha, \quad (s, \sigma) \in \delta_0.
\]

With the above considerations we may provide a finite dimensional linear programming relaxation of Problem 2 based on the sampled values of \( w(t, k) \) and \( \theta(t, k) \) at points of interest:

\[
\begin{aligned}
\omega_k &:= w(t_k, k) = e^{\delta_0 w_0}, \quad k \in \mathbb{K}, \\
\omega_h &:= w(s_h, \sigma_h) = e^{\delta_0 w_0}, \quad h \in \mathcal{H},
\end{aligned}
\]

Using the above notation, the linear programming problem of interest can be compactly defined as follows.

**Problem 3.** Find, if possible, \( \alpha^*, \theta_0^*, \ldots, \theta_{K-1}^* \) such that

\[
\alpha^* = \min_{\alpha, \theta_0, \ldots, \theta_{K-1}} \alpha \quad s.t.
\]

\[
-\alpha \leq G_\omega \omega_0 + G_w(\omega_0) \bar{\theta}_h \leq +\alpha, \quad h \in \mathcal{H},
\]

\[
0 = \Pi_{\omega}(\alpha_\omega) \delta_k, \quad k \in \mathbb{K},
\]

where \( \Pi_{\omega}(v) := \left[ \Pi_1 v \Pi_2 v \cdots \Pi_T v \right] \) satisfies \( \Pi_{\omega}(v) \theta = \Pi(\theta) v \).

**Remark 2.** Although problem (23a) is always feasible (choose \( \theta_k = 0, k \in \mathbb{K} \), and \( \alpha = \max |G_w(\omega_0)|, h \in \mathcal{H}, \) it can have multiple and possibly unbounded minimizers. Since in practice large values of \( \theta_k \) (and \( \delta_k \)) as well as multiple solutions might cause implementation issues, it can be often desirable to introduce in (23a) additional quadratic terms yielding

\[
\min_{\alpha, \theta_0, \ldots, \theta_{K-1}} \alpha^2 + \epsilon_1 \sum_{k=0}^{K-1} \delta_k^2 + \epsilon_2 \sum_{k=0}^{K-1} \delta_k^2 \delta_k
\]

with \( \epsilon_1, \epsilon_2 \geq 0 \), such that \( \epsilon_1 > 0 (\epsilon_2 > 0) \) penalizes large values of \( \theta_k (\delta_k) \), and any choice such that \( \epsilon_1 > 0 \) implies a unique, bounded optimizer (due to strict convexity). Even if it is no longer linear, the resulting convex optimization problem can still be efficiently solved. In addition, or as an alternative, it is also possible to introduce explicit bounds on \( \theta_k \) and \( \delta_k \), like \( \underline{\theta} \leq \theta_k \leq \bar{\theta}, \quad k \in \mathbb{K} \). It is also worth of mentioning that by simply choosing \( K = 1 \) and considering the periodicity constraint, the same formulation yields an optimal constant selection \( \theta(t, k) = \overline{\theta} \) for all \( (t, k) \in T \), which is often a preferred option for its simpler implementation, and corresponds to the solution proposed in our preliminary work (Galeani, Serrani, Varano, & Zaccarian, 2011).

Define \( |\delta_0| := \max(s_{h+1} - s_h, h \in \mathcal{H}) \). Let \( \alpha_{T}^{ip}, \theta_{k}^{ip} \) and \( \alpha_{T+\mathcal{E}}, \theta_{k+\mathcal{E}}^{ip} \) denote, respectively, the optimal solutions to Problems 3 and 2. Finally, let \( \theta^{ip}(\theta^*) \) denote (20) with \( \theta_k = \theta_{k+\mathcal{E}}^{ip} \). The following proposition clarifies that an arbitrarily close approximation of the optimal solution to the original Problem 2 (equivalently, Problem 1) for the class \( \theta_T \) considered in (28) can be obtained by solving Problem 3 for a sufficiently small choice of \( |\delta_0| \).
Proposition 3. For any \( \varepsilon > 0 \), there exists \( \delta = \delta(\varepsilon, w_0) > 0 \) such that if \( |\delta_0| < \delta \) then
\[
\alpha^* + \varepsilon \leq j(\theta^*, w_0) \leq \alpha^* + \varepsilon,
\]
(25a)
\[
\alpha^* - \varepsilon \leq j(\theta^*, w_0) \leq \alpha^* + \varepsilon.
\]
(25b)

Moreover, if \( w_0 \) is known to belong to a compact set \( W_0 \), then \( \delta = \delta(\varepsilon) > 0 \) can be chosen independent of \( w_0 \).

**Proof.** The proof is only sketched, since it is based on standard arguments. First, note that defining
\[
J_\delta(\theta, w_0) = \min \alpha \text{ s.t. } (22), \quad (18a)
\]
as a sampled version (on \( \delta_0 \)) of
\[
J(\theta, w_0) = \min \alpha \text{ s.t. } (19), \quad (18a)
\]
and since \( \delta_0 \subseteq T_0 \), it follows that, for any \( \theta(\cdot) \),
\[
J_\delta(\theta, w_0) \leq J(\theta, w_0).
\]
(26)

The first inequality in (25a) follows directly from (26), since then \( \alpha^* + \varepsilon \leq J(\theta^*, w_0) \). The first inequality in (25b) follows by contradiction from (26), since assuming \( J(\theta^*, w_0) < \alpha^* + \varepsilon \) would also imply \( J_\delta(\theta, w_0) < \alpha^* + \varepsilon \) \( \equiv \) \( \min \alpha \), \( (\theta, w_0) \). The second inequality in (25b) follows by contradiction from the second inequality in (25a), since assuming \( J(\theta^*, w_0) > \alpha^* + \varepsilon \) would imply \( J(\theta^*, w_0) \leq \alpha^* + \varepsilon \). In order to prove the second inequality in (25a), consider any \( (t, k) \in T_0 \) and \( (\theta, w(t,k)) \in \mathcal{F} \) such that \( \sigma_0 \) and \( \sigma_{k+1} \) are such that \( \mathcal{F} = \mathcal{F}(\theta^*) \) and \( \bar{w}(t,k) \) is given by the constraint (23b). Hence, it is enough to show that there exists a \( \delta(\varepsilon, w_0) \) such that for all \( \delta(\varepsilon, w_0) \)
\[
\begin{align*}
|u_\delta(\theta^* + \varepsilon, w(t,k)) - u_\delta(\theta^* - \varepsilon, \bar{w}(t,k))| & \leq \varepsilon.
\end{align*}
\]
(27)

Let \( \bar{\delta} = t - \sigma_0 \), which by the hypothesis on \( \delta_0 \) satisfies \( \bar{\delta} \leq |\delta_0| < \delta \).

Let \( w(t,k) = \bar{e}^\delta \sigma_0 \), the argument on the left hand side of (27) can be rewritten as
\[
\begin{align*}
|u_\delta(\theta^* - \varepsilon, w(t,k)) - u_\delta(\theta^* + \varepsilon, \bar{w}(t,k))| &= \left| \left( \Gamma_p + \Gamma_\theta(\theta^* + \varepsilon) \right) \left( e^{\delta} - 1 \right) \right| |ar{w}(t,k)|
\end{align*}
\]
(28)

Since \( \lim_{\delta \to 0} \left( e^{\delta} - 1 \right) = 0 \), it follows that for sufficiently small \( \delta(\varepsilon, w_0) \) the whole quantity can be made as small as desired, and then it is possible to guarantee (27). Finally, proving that allowing \( w_0 \) to range in a compact set \( W_0 \) implies that \( \delta \) can be chosen independent from \( w_0 \) involves standard arguments, whose details are omitted for brevity. \( \square \)

5. A simulation example

Consider the system described by the following matrices:
\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\]
\[
= \begin{bmatrix}
0.7 & -0.8 & -0.7 & -0.8 & 0.3 & 0.5 & 0.4 & 0.7 \\
0.9 & -0.5 & 1.0 & -0.2 & -1.0 & 0.5 & -1.0 & 0.4 \\
-0.8 & 0.1 & 1.0 & 0.9 & 0.7 & -0.2 & -0.5 & 0.4 \\
0.9 & 1.0 & 0.0 & 0.6 & 0.9 & 0.3 & -1.0 & 0.9 \\
0.3 & 1.0 & 0.6 & 1.0 & 0.4 & -0.7 & -0.8 & 1.0 \\
-0.1 & 0.6 & -0.7 & -0.1 & 0.4 & 0.0 & 0.0 & 0.0 \\
-0.2 & 0.6 & 0.0 & 0.3 & 0.5 & 0.0 & 0.0 & 0.0
\end{bmatrix}
\]

which have been generated randomly, except for the matrix \( S \). This latter one was specifically selected in order to satisfy Assumption 3 (requiring periodicity of the exosystem’s free responses) with period \( P = 1 \). Assumptions 1 and 2 are both satisfied, so the results of Theorem 1 can be applied. The dimension of the free parameter \( \theta \) is given by \( s = (m - p)q = (3 - 2)7 = 7 \). For the initial state of the exosystem chosen as \( w_0 = [2 1 0 1 0 1 0 \; 0] \), and a bound on each element of \( \theta \) to be less than 100 in magnitude, the achieved performance levels (in terms of infinity norm of the steady state input) are listed in Table 1. From this data, it is evident that using an optimal constant value of \( \theta \) (instead of just squaring down the plant by removing inputs) allows for a strong reduction of the steady-state input magnitude. Further reductions can be achieved by using a time-varying choice of \( \theta \), and increasing the number of jumps in one period (in the piecewise constant case). It is interesting to notice that additional advantages might be achievable by removing the steady-state condition (14), but exploitation of such quasi steady-state solution requires the development of a suitable definition of the corresponding hybrid steady-state solution along the lines of Carnevale, Galeani, and Menini (2012), Cox, Marconi, and Teel (2014) and Marconi and Teel (2013), which will be treated in future work.
Simulations have been performed considering the case of a piecewise constant $\theta(\cdot)$ having 10 equally spaced jumps during each period, and using as stabilizers $F$ an LQR state feedback gain (not a friend of $\mathcal{R}^*$) and a state feedback gain which is a friend of $\mathcal{R}^*$. Figs. 2 and 3 show the resulting inputs (both in steady state and for a complete response, starting from $x(0) = 0$) and regulation errors. In particular, Fig. 2 shows how choosing an optimal $\theta$ (in the considered class) by way of the algorithm proposed in Section 4 allows for a magnitude reduction of the steady-state input with respect to the simpler choice of suppressing one input in order to impose a unique solution of the regulator equations (7). In turn, Fig. 3 shows how, although case (1) of Theorem 1 under (14) yields regulation error convergence even when using a feedback stabilizer which is not a friend of $\mathcal{R}^*$, Proposition 2 ensures that if a friend of $\mathcal{R}^*$ is used then the motion of $\theta$ will not affect $e$ at all (whereas in the small box of Fig. 3 it is evident that the jumps in $\theta$ affect $e$ when using a gain which is not a friend of $\mathcal{R}^*$). Interestingly, Fig. 3 illustrates that the friend stabilizer leads to a reduced peaking of the regulation error (upper plot), in spite of a larger peaking of $|\dot{x}|$ (lower plot). Fig. 4 illustrates what happens in case (2) of Theorem 1, when (14) is not imposed. In such a case, Theorem 1 (based on the properties in Proposition 2) still yields convergence to zero of the regulation error (Fig. 4, top), despite the fact that the variations in $\theta$ produce large differences between the actual plant state response and the quasi steady-state response in the state of the plant (Fig. 4, bottom).

6. Conclusions

The full-information regulator problem for over-actuated LTI systems has been investigated in this paper. By focusing on the relevant case of systems with full-rank input operators, the properties of the solution to the regulator equations have been exploited to parameterize the redundancy in the steady state trajectories that are compatible with the error-zeroing condition. This parameterization is exploited toward the minimization of appropriate functionals. The most general structure of the proposed allocation mechanism, which takes the form of a hybrid system, leaves open important questions regarding the correct interpretation of “hybrid steady-state solutions”, which have been only slightly touched upon in this paper. These issues, as well as the extension to larger classes of output regulation problems, are the subject of current investigation.

Appendix A. A useful decomposition

Let $\rho := \dim \mathcal{R}^*$ and $\nu := \dim \mathcal{V}^*$, where $\nu \geq \rho$ since $\mathcal{R}^* \subset \mathcal{V}^*$. Choose $T \in \mathbb{R}^{n \times n}$ as an invertible matrix such that its first $\rho$ columns span $\mathcal{R}^*$ and its first $\nu$ columns span $\mathcal{V}^*$. Choose $G = [G_1, G_2, G_3] \in \mathbb{R}^{m \times n}$ as an invertible matrix such that $\ker G_1 = B^{-1} \mathcal{R}^* \cap \ker D$ and $\ker G_2 = \ker D$. Choose $M = [M_1, M_2] \in \mathbb{R}^{p \times n}$ as an invertible matrix such that $M_1 D G_3 = 0$, $M_2 D G_3 = I$. Choose $F \in \mathbb{R}^{m \times n}$, $F = F_0 + F_1$, with $F_1$ being a friend of $\mathcal{V}^*$ and $F_0 = -G_3 M (C + D F_0^*)$ (so that also $F$ is a friend of $\mathcal{V}^*$). Apply the regular feedback transformation $u = F x + G u$, the output coordinate change $\tilde{e} = Me$ and the coordinate change $z = T^{-1} x$ to system $\mathcal{P}$ to obtain

$$
\dot{z} = \tilde{A}^T z + \tilde{B} u + P w, \quad \tilde{e} = \tilde{C}^T z + \tilde{D} u + \tilde{Q} w, \quad (A.1)
$$

where $\tilde{A}^T = T^{-1} (A + BF) T$, $\tilde{B} = T^{-1} B G$, $\tilde{P} = T^{-1} P$, $\tilde{C}^T = M (C + D F)^T$, and $\tilde{D} = M D G$ and $\tilde{Q} = M Q$, having the structure

$$
\begin{bmatrix}
\tilde{A}^T & \tilde{B} \\
\tilde{C}^T & \tilde{D} \\
\end{bmatrix}
= \begin{bmatrix}
\tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} & \tilde{B}_{11} & \tilde{B}_{12} & \tilde{B}_{13} & \tilde{P}_1 \\
0 & \tilde{A}_{22} & \tilde{A}_{23} & 0 & \tilde{B}_{22} & \tilde{B}_{23} & \tilde{P}_2 \\
0 & 0 & \tilde{A}_{33} & 0 & \tilde{B}_{33} & \tilde{B}_{33} & \tilde{P}_3 \\
0 & 0 & 0 & 0 & 0 & 0 & \tilde{Q}_1 \\
0 & 0 & 0 & 0 & 0 & 0 & \tilde{Q}_2 \\
\end{bmatrix}
$$

where, in particular, $\tilde{A}_{11} \in \mathbb{R}^{p \times p}$, $\tilde{A}_{22} \in \mathbb{R}^{(\nu-\rho) \times (\nu-\rho)}$, $\tilde{A}_{33} \in \mathbb{R}^{(\nu-\nu) \times (\nu-\nu)}$, and $\tilde{B}_{11} \in \mathbb{R}^{p \times p}$. Recall that, by definition, the pair $(\tilde{A}_{11}, \tilde{B}_{11})$ is controllable, hence the spectrum of $\tilde{A}_{11}$ is assignable by a friend of $\mathcal{R}^*$ (hence, by a friend of $\mathcal{V}^*$) (Trentelmaan et al., 2001, Th. 4.18). Conversely, spec $\tilde{A}_{22}$ is the set of all eigenvalues of $\tilde{A}_{22}$, that cannot be assigned by a friend of $\mathcal{V}^*$, which coincides with the set of invariant zeros of $(A, B, C, D)$. In particular matrix $\tilde{A}_{22}$ represents, in the given coordinates, the map induced on $\mathcal{V}^*/\mathcal{R}^*$ by $A$ (Th. 7.14 Trentelmaan et al., 2001). Note that under item 2 of Assumption 1, the pair $[\tilde{B}_{22} \tilde{B}_{23} \tilde{B}_{32} \tilde{B}_{33}]$, is stabilizable, as it can be easily verified by the PBH test.
Lemma 1. Under Assumption 2, the subsystem described by the quadruple \((A_{1}, B_{1}, C_{1}, 0)\) above is square and has no finite invariant zeros.

Proof of Lemma 1. The system matrix \(\tilde{P}_{2}(s)\) of (A.1) is related to the original \(P_{2}(s)\) of (1) by pre- and post-multiplication by constant invertible matrices
\[
\begin{bmatrix}
\tilde{A} & \tilde{B} \\
\tilde{C} & \tilde{D}
\end{bmatrix} = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix} \begin{bmatrix}
\gamma^{-1} & 0 \\
0 & M
\end{bmatrix} \begin{bmatrix}
\tilde{A} & \tilde{B} \\
\tilde{C} & \tilde{D}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\gamma & 0 \\
0 & C
\end{bmatrix},
\]
hence \(\text{rank} \; \tilde{P}_{2}(s) = \text{rank} \; P_{2}(s)\) as polynomial matrices. Considering the structure of the matrices \((\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})\) and rearranging the columns of \(\tilde{P}_{2}(s)\) yields the matrix
\[
\begin{bmatrix}
\tilde{A}_{11} - sI & \tilde{B}_{11} \\
0 & \tilde{A}_{22} - sI & \tilde{B}_{21} & \tilde{B}_{22}
\end{bmatrix},
\]
which by Assumption 2 (and the fact that only invertible transformations were used) must still have rank \(n + p\) as a polynomial matrix. This imposes that the block involving \(\tilde{A}_{11}, \tilde{B}_{11}\) and \(\tilde{C}_{11}\) is full row rank, so that \(\tilde{B}_{32}\) must have at least \(\rho_{1} - \rho_{3}\) independent columns, where \(\rho_{1}\) is the number of columns of \(\tilde{B}_{11}\). In fact, \(\tilde{B}_{32}\) has exactly \(\rho_{1} - \rho_{3}\) columns, as shown next. Suppose that \(\tilde{B}_{32}\) contains more than \(\rho_{1} - \rho_{3}\) columns. For any \(s = \bar{s}_{0} \in \Re\) there would exist \(z_{30}, \bar{u}_{20}\) such that
\[
\begin{bmatrix}
\tilde{A}_{11} - s\bar{s}_{0}I & \tilde{B}_{11} \\
\bar{u}_{3} & \bar{u}_{20}
\end{bmatrix} \begin{bmatrix}
\bar{z}_{30} \\
\bar{u}_{20}
\end{bmatrix} = 0,
\]
so that defining \(z_{a} := \begin{bmatrix} 0 & 0 & \bar{z}_{30} \end{bmatrix}^T\), \(z_{a} \notin \mathcal{V}^*\), \(\bar{u}_{a} := \begin{bmatrix} 0 & \bar{u}_{1} & \bar{u}_{20} & 0 \end{bmatrix}^T\) and \(\mathcal{V}^* := \text{span}(z_{a}) + \mathcal{V}^*\), one has:
\[
\begin{bmatrix}
\bar{A}^T \\
\bar{C}^T
\end{bmatrix} z_{a} = \begin{bmatrix}
\bar{A}_{11}z_{30} \\
\bar{A}_{12}z_{30} \\
\bar{A}_{22}z_{30} \\
0
\end{bmatrix} = \begin{bmatrix}
\bar{B}^T & \bar{D}
\end{bmatrix} \bar{u}_{a},
\]
where the first two terms belong to \(\mathcal{V}^* \subset \mathcal{W}^*\), the third belongs to \(\mathcal{W}^*\) and the last is in \(\mathcal{B}^T \mathcal{D}^T \mathcal{F}\) implying that the subspace \(\mathcal{W}^*\) is strictly larger than \(\mathcal{V}^*\) and satisfies (2); but this contradicts the maximality of \(\mathcal{V}^*\). A similar reasoning applies for \(s = \bar{s}_{0} \in \Re \setminus \Re\) by considering the subspace generated by the real and the imaginary parts of \(z_{a}\), which implies that there are no finite invariant zeros. □

Appendix B. Proof of the main results

Proof of Proposition 1. The regulator equations (7) can be cast as the Hautus equation (Hautus, 1983) \[A_{1}X - A_{2}XS = E\] by defining \[A_{1} := \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad A_{2} := \begin{bmatrix} I \end{bmatrix}, \quad E := \begin{bmatrix} P \end{bmatrix}.\] Let \(\mathcal{K} : \Re^{m+n+q} \rightarrow \Re^{(m+p+q)}\) be the linear operator such that \(\mathcal{K}(X) := A_{1}X - A_{2}XS\). The set of all solutions to the Hautus equation \(\mathcal{K}(X) = E\) above is given by the linear variety \(\mathcal{V} = X_{p} + \text{ker} \; \mathcal{K}, \; X_{p} = \{P_{p}, \; \Gamma_{p}\}\) is a particular solution. It has been shown in Hautus (1983) that the operator \(\mathcal{K}\) is surjective if and only if \(\text{rank} \; (A_{1} - A_{2}\lambda) = n + p\), for all \(\lambda \in \text{spec} \; S\), which implies by Assumption 2. Consequently, \(\dim \text{ker} \; \mathcal{K} = (m - p)\gamma = s\), consistently with the fact that the solution to (7) is unique if \(m = p\). If \(s > 0\) it follows that all solutions to (7) are given by (9). □

Proof of Proposition 2. With reference to the proof of Proposition 1, denote by \(\mathcal{K}\) the space of all solutions to the homogeneous equation (8). It is easy to see that \(\mathcal{K}\) is invariant with respect to regular feedback transformations (and, obviously, coordinate transformations). In particular, \(X := (F; \; F') = (F^{-1}P; \; C^{-1}G - G^{-1}F P')\) solves
\[
\mathcal{K}(X) = \mathcal{K}(F; \; F'), \quad 0 = \mathcal{K}(F; \; F'),
\]
which are the homogeneous regulator equations associated to the transformed system (A.1). Consequently, it is enough to prove the proposition for system (A.1). This amounts to showing that \(\Pi_{a} = 0\) and \(\Pi_{a} = 0\), where \(\Pi = \{\Pi_{1}; \; \Pi_{2}; \; \Pi_{3}\}\) and \(\Gamma = \{\Gamma_{1}; \; \Gamma_{2}; \; \Gamma_{3}\}\) are partitioned according to the given decomposition of system (A.1). The homogeneous regulator equations (B.1), in the new coordinates read as a set of four equations
\[
\Pi_{1}\Sigma = \bar{A}_{11}
\]

References


