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Trajectory tracking for a particle in elliptical billiards

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An infinitely rigid unitary mass (particle) is considered, moving on a planar region delimited by a rigid elliptical barrier (elliptical billiards) under the action of proper control forces. A class of periodic trajectories, involving an infinite sequence of non-smooth impacts between the mass and the barrier at fixed times, is found by using an LMIs based procedure. The jumps in the velocities at the impact times render difficult (if not impossible) to obtain the classical stability and attractivity properties for the dynamic system describing the tracking error behaviour. Hence, the tracking control problem is properly stated using notions similar to the quasi stability concept in V. Lakshmikantham, D.D. Bainov and P.S. Simeonov, Theory of Impulsive Differential Equations, 6, World Scientific, 1989. A controller (whose state is subject to discontinuities) based on the internal model principle is shown to solve the proposed tracking problem, giving rise to control forces that are piecewise continuous function of time, with discontinuities at the desired impact times and at the impact times of the particle with the barrier.

1. Introduction

Control problems for systems subject to impacts (Brogliaito 1996) are of interest in a variety of applications, especially in robotics, e.g., hopping (Schwind and Koditschek 1995) or walking robots (Morris and Grizzle 2005, Hurmuzlu et al. 2004, Westervelt et al. 2003), juggling robots (Buehler et al. 1994, Brogliaito and Zavala-Rio 2000), hammering tasks (Izumi and Hitaka 1997). Trajectory tracking for dynamical complementary systems (see Heemels and Brogliaito (2003) for an excellent overview on modelling, analysis and control for such systems), of which the class of mechanical systems subject to jumps is a subset, has been tackled recently in Brogliato et al. (1997, 2000), Bourgeot and Brogliato (2005), Pagilla and Yu (2001, 2004). More precisely, in Brogliato et al. (1997, 2000), Bourgeot and Brogliato (2005) a control scheme based on “classical” non-linear controllers (like passivity-based, etc.), adapted to the non-smoothness of the problem, that ensures stable tracking of some reference trajectories is presented. On the other hand, in Pagilla and Yu (2001, 2004) an event-based control-switching strategy is proposed which includes a stable discontinuous controller for the transition phases. In Menini and Tornambè (2001) PD-like control inputs are used to asymptotically stabilize particular periodic trajectories for a planar mechanical system within a Birkhoff curvilinear billiard. In this paper a dimensionless body of unitary mass (particle), which moves on a plane, in a convex region delimited by an elliptical barrier assumed to be infinitely massive and rigid is considered; the particle, when uncontrolled, reflects perfectly on the boundary and follows a straight-line path between two impacts. Such a system is called an elliptical billiards.

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The notion of billiard system was introduced by Birkhoff (1927), and from then on it became a challenging research field, which has attracted the attention of researchers from mathematics, engineering and physics, where billiards are used to investigate the transition from quantum mechanics to classical mechanics (Wiersig 2001). A lot of work has been done to study the properties of trajectories followed by a free particle on billiards of different shape, that is when no control is exerted on the moving mass (Kozlov and Treshchev 1991). However, for their particular dynamical features, billiards are a very interesting benchmark for studying many (impact) control problems. In Sepulchre and Gerard (2003) and Gerard and Sepulchre (2004) stabilization results for periodic orbits of the controlled wedge billiard are obtained. In Menini and Tornambè (2001) an actuated particle moving in a circular billiards was considered having as a control goal the tracking of a class of periodic trajectories. In the present work, an actuated particle is considered again and closed polygons inscribed in an ellipse (centered at the origin with \( a \) and \( b \) being the semi-major and the semi-minor axis, respectively) and having vertices on the ellipse are considered as reference paths. Analytic conditions for the closure of polygons in elliptical billiards are obtained by Cayley in the mid-19th century (see Cayley (1854), but also Lebesgue (1942), Griffiths and Harris (1978) and Berger (1987)). The higher-dimensional generalization of these conditions have been recently obtained in Dragovic and Radnovic (1998a, b). See also Dragovic and Radnovic (2006) for generalizations in several different directions. In order to find velocity profiles that yield resulting periodic trajectories admissible for the controlled billiard system with the impact times being equally spaced, a motion planning problem is properly stated and solved in the first part of the paper. In particular, it is shown that when the velocity profiles are generic polynomials of degree \( q \), such a problem can be addressed through LMIs by using some results from the theory of non-negative polynomials (Nesterov 2000, Henrion and Lasserre 2006). An easy algorithm is given here for finding such a class of polynomials when the problem parameters are fixed. In the second part of the paper, a tracking control problem is formulated where the goal is to asymptotically track one of the trajectories specified above. Since the vector state of the considered dynamical system is intrinsically subject to jumps, the classical stability and attractivity properties cannot be obtained (as shown in Menini and Tornambè (2001)) so that suitable modifications are required. Hence, the tracking control problem dealt with in this paper is quite similar to the one in Menini and Tornambè (2001), where notions similar to the quasi stability concept proposed in Lakshmikantham et al. (1989) for impulsive differential systems are used. In order to solve the proposed control problem a controller based on the internal model principle has been designed. It is to be stressed that, similarly to the system to be controlled, the state of the precompensator used to guarantee the presence of the internal model of the reference trajectory presents discontinuities.

All the material concerning the solution of the motion planning problem is completely new with respect to Menini and Tornambè (2001). As a matter of fact, in the aforementioned paper, being the billiards a circular one and the desired trajectories regular polygons with constant velocity, the desired trajectories were simply a class of trajectories for the open loop system, whereas here, being the billiards an elliptical one and still requiring that the desired trajectories have jumps at equally spaced times, it is needed, in general, that a proper control action is delivered to the particle in order to make the desired trajectory admissible for the controlled billiard system. Moreover, also the algorithm proposed in this paper to solve the tracking control problem is different from the one proposed in Menini and Tornambè (2001), where a simple PD control law was used. As a consequence, the proofs reported here are dissimilar to those of Menini and Tornambè (2001): there, in view of the simplicity of the closed-loop system, it was possible to compute some expressions in closed form and use them for a direct proof of many facts, whereas here a more abstract setting has been preferred to avoid some extremely cumbersome computations.

The outline of the paper is as follows. In §2, after some preliminary notation and definitions, the equations of motion of a particle within an elliptical billiards are given. The problem to find a class of periodic trajectories, admissible for the controlled billiard system under some imposed constraints, is analysed in §3. In §4, the asymptotic tracking problem in presence of non-smooth impacts is stated and solved using a control algorithm based on the internal model principle. Finally, simulation examples to show the effectiveness of the proposed control law and some concluding remarks are given in §5 and 6, respectively. In Appendix A a detailed version of the LMI problem solving the motion planning problem stated in §3 is reported and the details of the proof of the solution to the control problem stated in §4 are given in Appendix B.

2. Problem preliminaries

In the following, \( \mathbb{R} \) will denote the set of real numbers, \( \mathbb{R}^+ \) the set of non-negative real numbers, \( \mathbb{Z} \) the set of integers, \( \mathbb{Z}^+ \) the set of non-negative integers, \( \mathbb{N} \) the set
of positive integers, \( \mathbb{R}^n \) the set of vectors of dimension \( n \), 
\( \mathbb{R}^{n \times n} \) the set of real matrices of dimensions \( m \times n \), 
with \( m, n \in \mathbb{N} \). For the sake of brevity, the shorthand notations \( g(r^{-}) \) and \( g(r^{+}) \) will be used in place of 
\( \lim_{r \to r^{-}} g(t) \) and \( \lim_{r \to r^{+}} g(t) \), respectively, for any 
vector function \( g(t) \). Finally, \( I \) and \( 0 \) will denote the identity matrix 
and the zero matrix, respectively, whose size will be clear from context.

**JordanBlock** \((v, \lambda)\) a Jordan block of size \( v \) associated 
with the eigenvalue \( \lambda \), i.e., a matrix of dimensions 
\( v \times v \) which is composed of 0 everywhere except for 
the main diagonal, which is filled with \( \lambda \), and for 
the superdiagonal, which is composed of 1, the symbol 
\( \| \cdot \| \) the Euclidean norm of the vector at argument 
and the symbols \( [t] \) and \( (t) \) the largest integer smaller 
than or equal to \( t \) and the integer nearest to \( t \), 
respectively. In the case in which \( t \) is a half-integer, \( (t) \) 
denotes the smallest integer larger than \( t \). For the 
sake of brevity time dependence will often be omitted, 
whenever clear from the context.

Consider a dimensionless body having unitary mass 
(particle), which moves on an horizontal plane, on 
which a Cartesian reference \( xOy \) is defined. Let 
\( \mathbf{q}(t) = [x(t), y(t)]^T \in \mathbb{R}^2 \) denotes the position of the 
body at time \( t > t_0 \), with \( t_0 \in \mathbb{R}^+ \), and let \( \mathbf{q}(t) \) be 
constrained to belong to the following admissible region:

\[
A := \{ \mathbf{q} \in \mathbb{R}^2 : f(\mathbf{q}) \leq 0 \}, \quad \text{where} \quad f(\mathbf{q}) := \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1, \quad a, b \in \mathbb{R}^+, \quad a > b,
\]

that is \( f(\mathbf{q}) = 0 \) defines an ellipse centered at the 
origin with \( a \) and \( b \) being the semi-major and semi-
minor axes, respectively. Let the control inputs be 
two forces \( u_s(t), u_r(t) \in \mathbb{R} \), acting directly on the 
considered body, having directions parallel to 
the \( x \) and \( y \) axes, respectively; \( u_s(t) \) and \( u_r(t) \) are positive 
when directed as the \( x \) and \( y \) axes, respectively. It is 
also assumed that \( \mathbf{q}(t) \) and \( \dot{\mathbf{q}}(t) \) are measured.

The system is completely characterized by the 
Lagrangian function 
\( L := (1/2)(\dot{x}^2 + \dot{y}^2) + u_s x + u_r y \), 
by the inequality \( x^2/a^2 + y^2/b^2 \leq 1 \) and by the assumption 
that the impacts are non-smooth, perfectly elastic 
and without friction. The method of the Valentine 
variables is used for modelling the considered mechanical 
system as in Tornambè (1999). One real-valued 
Valentine variable \( \gamma(t) \) is introduced so that the inequality 
constraint characterizing the admissible region is 
transformed into the equality constraint 
\( f(\mathbf{q}(t)) + \gamma^2(t) = 0 \). Since \( \gamma(t) \) is taken real, such an 
equality constraint is completely equivalent to the 
original inequality constraint. Taking the derivative 
with respect to time of both sides of the equality 
constraint, the differential constraint 
\( (2/a^2)\dot{x}(t)x(t) + 
(2/b^2)\dot{y}(t)y(t) + 2\gamma(t)\dot{\gamma}(t) = 0 \) is obtained. Starting from 
the initial conditions \( \mathbf{q}(t_0) \in A \) and \( \gamma(t_0) = \sqrt{-f(\mathbf{q}(t_0))} \), 
the differential constraint is completely equivalent to 
the equality constraint.

The actual path of motion can be found by looking 
for the stationary value of the unconstrained functional 
\( \int_{t_1}^{t_2} \dot{L} \, dt \), where \( \dot{L} := L + \lambda((2/a^2)\dot{x}\ddot{x} + (2/b^2)\dot{y}\ddot{y}) + 2\gamma(t)\dot{\gamma}(t) \), \( \lambda \) is a Lagrange multiplier and \( [t_1, t_2] \) is the time interval over which the motion is studied. The stationary value of 
the unconstrained functional corresponds to the 
path of motion that is solution (in each open interval 
of time without impacts) of the following Euler-
Lagrange equations:

\[
\begin{align*}
\ddot{x}(t) + \frac{2}{a^2} \lambda(t) x(t) &= u_s(t), \\
\ddot{y}(t) + \frac{2}{b^2} \lambda(t) y(t) &= u_r(t), \\
2\gamma(t)\dot{\gamma}(t) &= 0, \\
\frac{2}{a^2} \dot{x}(t)\dot{x}(t) + \frac{2}{b^2} \dot{y}(t)\dot{y}(t) + 2\gamma(t)\dot{\gamma}(t) &= 0,
\end{align*}
\]

where \( \dot{\lambda}(t) \) has to be understood in the distributional 
ssence and the terms \( (2/a^2)\dot{x}\ddot{x} \) and \( (2/b^2)\dot{y}\ddot{y} \) represent 
the reaction forces exchanged between the mass 
and the elliptical barrier, along the \( x \) and \( y \) directions, 
respectively (which, of course, are equal to zero 
whenever the mass is not in contact with the barrier).

The impacts occur only at times \( t_i \in \mathbb{R}, i \in \mathbb{N} \), 
where the following Erdmann–Weierstrass corner 
conditions (see, e.g., Luenberger (1989), Bliss (1946) 
and Wan (1995)), which are necessary at corner points 
where \( \mathbf{q}(t) \) is not differentiable, are satisfied

\[
\begin{align*}
\dot{x}(t_i^-) + \frac{2}{a^2} \lambda(t_i^-) x(t_i^-) &= \dot{x}(t_i^+) + \frac{2}{a^2} \lambda(t_i^+) x(t_i^+), \\
\dot{y}(t_i^-) + \frac{2}{b^2} \lambda(t_i^-) y(t_i^-) &= \dot{y}(t_i^+) + \frac{2}{b^2} \lambda(t_i^+) y(t_i^+).
\end{align*}
\]
The initial conditions at the initial time $t_0$ are

\[
\begin{align*}
  x(t_0) &= x_0, \quad y(t_0) = y_0, \\
  \dot{x}(t_0) &= v_{x,0}, \quad \dot{y}(t_0) = v_{y,0}, \\
  y(t_0) &= \sqrt{-f(q(t_0))}, \quad \dot{\lambda}(t_0) = 0.
\end{align*}
\]

For later use, the following set is introduced:

\[
\hat{A} := \{ (q, \dot{q}) \in \mathcal{A} \times \mathbb{R}^2 : J(q) \dot{q} \leq 0 \text{ if } f(q) = 0 \},
\]

where $J(q) = [(2/a^2)x, (2/b^2)y]$. For the initial time $t_0$, it is required that $(q(t_0), \dot{q}(t_0)) \in \hat{A}$ so that, if $q(t_0)$ is on the boundary, the velocity vector $\dot{q}(t_0)$ points toward the interior of the admissible region.

An impact for the controlled body occurs if, at a given time $t_i > t_0$, one has

\[
f(q(t_i)) = 0 \quad \text{and} \quad J(q(t_i)) \dot{q}(t_i) > 0.
\]

By requiring that $J(q(t_i)) \dot{q}(t_i) \leq 0$, the Erdmann–Weierstrass corner conditions (2) can be solved uniquely in the unknowns $\dot{x}(t_i), \dot{y}(t_i)$ and $\dot{\lambda}(t_i)$ at an impact time $t_i$ as

\[
\begin{align*}
  \dot{x}(t_i) &= C_1(q(t_i)) \dot{x}(t_i) + C_2(q(t_i)) \dot{y}(t_i), \quad (3a) \\
  \dot{y}(t_i) &= C_2(q(t_i)) \dot{x}(t_i) - C_1(q(t_i)) \dot{y}(t_i), \quad (3b)
\end{align*}
\]

where $C_1(q(t_i)) := (a^2 \sin^2(\theta_i) - b^2 \cos^2(\theta_i))/(a^2 \sin^2(\theta_i) + b^2 \cos^2(\theta_i))$ and $C_2(q(t_i)) := (a^2 \sin(2\theta_i))/(a^2 \sin^2(\theta_i) + b^2 \cos^2(\theta_i))$.

As for the Lagrange multiplier, its jump is given by

\[
\dot{\lambda}(t_i) = \lambda(t_i) + C_3(q(t_i)) \dot{x}(t_i) + C_4(q(t_i)) \dot{y}(t_i),
\]

where $C_3(q(t_i)) := (a^2 \sin(2\theta_i))/(a^2 \sin^2(\theta_i) + b^2 \cos^2(\theta_i))$ and $C_4(q(t_i)) := (ab \sin(2\theta_i))/(a^2 \sin^2(\theta_i) + b^2 \cos^2(\theta_i))$.

At an impact time $t_i$, the particle hits the elliptical boundary at point $q(t_i)$, so using the polar representation of an ellipse, $C_1(q(t_i))$ and $C_2(q(t_i))$ in (3) can be replaced by $\tilde{C}_1(\theta_i) := (a^2 \sin^2(\theta_i) - b^2 \cos^2(\theta_i))/(a^2 \sin^2(\theta_i) + b^2 \cos^2(\theta_i))$ and $\tilde{C}_2(\theta_i) := (ab \sin(2\theta_i))/(a^2 \sin^2(\theta_i) + b^2 \cos^2(\theta_i))$, respectively, where $\theta_i \in \mathbb{R}$ is the polar angle that the vector from the origin to point $q(t_i)$ makes with the positive direction of the $x$-axis (the polar angle is positive when measured counterclockwise).

3. A class of periodic trajectories

Classical studies of trajectories within a billiards assume that the particle moves with constant (say, unitary) velocity until it hits the boundary where, under the assumption of smooth boundary (e.g., elliptical), the mass reflects so that, with respect to the boundary, the tangential component of its velocity remains the same, while the normal component changes its sign.

In Menini and Tornambè (2001), the desired trajectories are constituted by regular polygons (having $N \geq 2$ vertices) inscribed in the circle of unitary radius centered at the origin and having one vertex coincident with the point $[1, 0]^T$; such a reference trajectory is the path followed by a not actuated (free) particle (with constant velocity between impacts) within the circular billiards and involving an impact at each time $t = i, i \in \mathbb{Z}$, if its initial position and velocity are properly chosen.

A trajectory is said to be “admissible” for the controlled billiard system if it belongs to the class of all possible trajectories corresponding to allowed control laws, i.e., piecewise continuous control laws without impulsive terms. This means that at the impact times the Erdmann–Weierstrass corner conditions (2) need to be satisfied. In the present work, polygons inscribed in an ellipse (centred at the origin with $a$ and $b$ being the semi-major and the semi-minor axis, respectively) having vertices on the elliptical barrier are considered as desired paths. In order to determine the desired (nominal) trajectory one of such paths has to be chosen and, as detailed in § 3.1, a velocity profile has to be designed such that each impact time $t_i$ occurs at an integer time.

Remark 1: The choice to impose impact times at arbitrarily fixed times permits to consider target periodic trajectories which are not admissible for the billiard system when no control is exerted on the moving mass. In this way, a more general problem is taken into account and the controller has to be designed such that an internal model of such a class of trajectories is contained in the closed-loop system, so that the control forces steer the tracking error to zero in the sense specified in the subsequent Problem 2.

The path described by a particle moving inside an elliptical billiards along straight-lines and reflecting from the boundary according to the reflection law is in general open, and it forms a dense subset of the region bounded by a caustic curve and the boundary (see, e.g., Crespi and Chang (1993)). A caustic is a curve inside the billiard such that if a segment (or its continuation) of a billiard path is tangent to it, then so is every other reflected segment (or its continuation) of the same path. In particular, for every path in an elliptical billiards, there exists a unique conic (the caustic) such that every segment (or its linear extension) of that path is tangent to such a conic (see, e.g., Jacobi (1884), Levi and Tabachnikov (2005) and Chernov and Markarian (2006)). If the initial segment of the billiard path falls outside the foci, then the caustic is an inner
ellipse (rotational motion); whereas, if the initial segment of the billiard path falls between the foci, then the caustic is a hyperbola (librational motion). Once a starting vertex $v_0$ is fixed on the elliptical boundary, then a path inside the billiards is completely determined by the choice of the caustic curve. In view of the Poncelet’s closure theorem (Poncelet 1822) the problem of finding periodic paths inside an elliptical billiards (closed Poncelet polygons) can be turned into the problem of finding particular caustics so that such paths close. Both kinds of motion can be characterized by the winding number $(N, R)$ with $N, R \in \mathbb{N}$. In particular, for the rotational motion, $N$ and $R$ denote the number of reflections and the rotation number (i.e., the number of circuits around the inner caustic) per period, respectively; whereas, for the librational motion, $N$ and $R$ denote the number of reflections and the libration number (i.e., half the number of touches at the inner caustic) per period, respectively.

In the next section, the problem to determine a velocity profile on the nominal path such that $t^d_i = i, i \in \mathbb{Z}$, and the corner conditions (3) hold at the impact times is stated and solved.

In summary, the desired trajectories considered in this paper are based on the classical closed polygonal orbits that were studied in the classical billiard theory (where no control is exerted on the particle), but the velocity profiles for the reference trajectories are of a more general kind.

### 3.1 Velocity profiles for the reference trajectories

Differently from the polygons in a circular billiards considered in Menini and Tornambè (2001), polygons inscribed in an elliptical billiards are in general non-regular and this implies that in general a solution at constant (scalar) velocity does not satisfy the requirements of having equally spaced impact times. Hence, in order to define a reference trajectory for the tracking control problem proposed in this note it is necessary to find a velocity profile such that the velocity along the desired path is chosen to satisfy some requirements and imposed constraints. More precisely, the following trajectory planning problem is considered.

**Problem 1**: Find velocity profiles such that for a particle moving along the desired paths (closed Poncelet polygons) impacts (in correspondence of polygon vertices) occur at each integer time, satisfying at such times the Erdmann–Weierstrass corner conditions (3).

To address this issue, define $\mathbf{q}_d(t) := [x_d(t), y_d(t)]^T$ as the position at time $t$ of a particle on the desired path. Requiring that each impact occurs at an integer time, that is $t^d_i = i, i \in \mathbb{Z}$, the desired trajectory can be represented as (in view of the periodicity of the desired path, the following relations hold $x_{d,i} = x_{d,(i \mod N)}$, $y_{d,i} = y_{d,(i \mod N)}$ and $l_i(\cdot) = l_{i(N)}(\cdot)$, for any integer $i$)

$$
\begin{bmatrix}
  x_d(t) \\
  y_d(t)
\end{bmatrix} = \begin{bmatrix}
  x_{d,[t]} \\
  y_{d,[t]}
\end{bmatrix} + l_{i(t-[t])} \begin{bmatrix}
  x_{d,[t]+1} - x_{d,[t]} \\
  y_{d,[t]+1} - y_{d,[t]}
\end{bmatrix},
$$

(4)

where $[x_{d,i}, y_{d,i}]^T, i \in \mathbb{Z}$ represents the position of the $i$th polygon vertex and $l_i(\cdot) : [0, 1] \to [0, 1]$, which determines the velocity profile on the trajectory segment from the vertex $i$ to the vertex $i+1$, has the following properties

$$
0 \leq l_i(\cdot) \leq 1 \quad \text{with} \quad \begin{cases} 
  l_i(0) = 0 \\
  l_i(1) = 1 
\end{cases}, \quad i \in \mathbb{Z}. \quad (5)
$$

In this paper, the case when the functions $l_i(s)$ are generic polynomials of degree $q \geq 1$ in $s := t - [t]$ is studied

$$
l_i(s) = a_q^i s^q + a_{q-1}^i s^{q-1} + \ldots + a_1^i s + a_0^i, \quad i \in \mathbb{Z}, \quad (6)
$$

where $a_j^i \in \mathbb{R}$ for each $j \in \{0, 1, \ldots, q\}$. Therefore, Problem 1 can be seen as the problem of finding such coefficients of $l_i(s)$ in order to satisfy (5) and the corner conditions (3). In particular, by substituting (4) into (3), at any impact time $i \in \mathbb{Z}$ the following relation has to be satisfied

$$
\dot{l}_i^N(t^d_i) \begin{bmatrix}
  x_{d,i+1} - x_{d,i} \\
  y_{d,i+1} - y_{d,i}
\end{bmatrix} = \dot{l}_i^N(t-[t]) \begin{bmatrix}
  C_1(q_i(\cdot)) & C_2(q_i(\cdot)) \\
  C_2(q_i(\cdot)) & -C_1(q_i(\cdot))
\end{bmatrix}
\begin{bmatrix}
  x_{d,i} - x_{d,i-1} \\
  y_{d,i} - y_{d,i-1}
\end{bmatrix}, \quad (7)
$$

where $C_1(\cdot)$ and $C_2(\cdot)$ are defined in (3) and $\dot{l}_i(t-[t])$ denotes the derivative with respect to time of $l_i(t-[t])$. Unfortunately, the system of equations and inequalities given by (5) and (7) is in general hard to solve. In spite of these difficulties, it is shown how the original problem can be restated so that the modified problem can be easily solved. As a matter of fact, it is easy to see that for any $i \in \mathbb{Z}$ guaranteeing $0 \leq l_i(s) \leq 1$ is completely equivalent to guarantee the non-negativeness of the following polynomials

$$
p_{1,i}(s) := l_i(s) \quad l_i(s) \geq 0 \iff p_{1,i}(s) \geq 0
$$

$$
p_{2,i}(s) := 1 - l_i(s) \quad l_i(s) \leq 1 \iff p_{2,i}(s) \geq 0. \quad (8)
$$

From the theory of non-negative polynomials (Nesterov 2000) a very important result states that a polynomial is non-negative if and only if it satisfies the
so-called sum-of-squares decomposition and the problem of satisfying such decomposition is reformulated as an LMI problem by the following result (see, e.g., Henrion and Lasserre (2006)).

**Lemma 1:** The polynomial non-negativity constraint

\[ p_j(s) = \sum_{j=0}^{q} c_j^i s^i \geq 0, \quad \forall s \in [s_i, s_j], \]

is equivalent to the existence of symmetric and positive semidefinite matrices \( Q_i \) and \( R_i \) of size \( m+1 \) with \( m = q/2 \) (q even) or \( m = (q-1)/2 \) (q odd), satisfying for any \( j = 0, 1, \ldots, q \) the convex LMI constraints

\[ c_j^i = \text{tr}(Q_i(H_{i-1} - s_i H_j)) + \text{tr}(R_i(s_j H_j - H_{j-1})), \]

where \( \text{tr}(X) \) denotes the trace of a generic square matrix \( X \) (i.e., the sum of its diagonal elements), and \( H_i \) is the Hankel matrix of dimension \( m+1 \) with ones along the \( (i+1)\)th anti-diagonal and zeros elsewhere.

Lemma 1 permits to reformulate the non-negativeness of polynomials \( p_i(s) \) and \( p_j(s) \) as an LMI problem, which can be solved using efficient solvers (Sturm 1999). In fact, considering the \( N \)-periodicity of the desired path, the inequalities in (8) are satisfied if and only if there exist symmetric and positive semidefinite matrices \( Q_i^i, R_i^i, Q_i^j, R_i^j \) for \( i = 0, \ldots, N-1 \) such that

\[
\begin{align*}
\alpha_0^i &= \text{tr}(R_i^i H_0), \\
\alpha_1^i &= \text{tr}(Q_i^i H_0) + \text{tr}(R_i^j(H_1 - H_0)), \\
\vdots \\
\alpha_q^i &= \text{tr}(Q_i^j H_{q-1}) + \text{tr}(R_i^j(H_q - H_{q-1})),
\end{align*}
\]

and

\[
\begin{align*}
\alpha_0^i &= 1 - \text{tr}(R_i^j H_0), \\
\alpha_1^i &= -\text{tr}(Q_i^j H_0) - \text{tr}(R_i^j(H_1 - H_0)), \\
\vdots \\
\alpha_q^i &= -\text{tr}(Q_i^j H_{q-1}) - \text{tr}(R_i^j(H_q - H_{q-1})).
\end{align*}
\]

Moreover, for any \( i \in \{0, 1, \ldots, N-1\} \) and \( j \in \{0, 1, \ldots, q\} \), the equations in (5) can be given in terms of the coefficients \( a_j^i \) as

\[
\begin{align*}
l_i(0) = 0 &\iff a_0^i = 0, \\
l_i(1) = 1 &\iff \sum_{j=0}^{q} a_j^i = 1,
\end{align*}
\]

whereas, regarding the corner conditions, relation (7) is turned into the following two equations

\[
\begin{align*}
a_j^i h_1(i) &= \left( \sum_{j=1}^{q} j a_j^{i-1} \right) h_2(i), \\
a_j^i w_1(i) &= \left( \sum_{j=1}^{q} j a_j^{i-1} \right) w_2(i),
\end{align*}
\]

Putting together and rearranging (9), (10), (11) and (12), an LMI problem to solve in the 4N decision variables \( Q_i^j, R_i^j, Q_i^i, R_i^i \) for \( i = 0, \ldots, N-1 \) is obtained (Appendix A contains a detailed version of such an LMI problem). More precisely, a solution of Problem 1 is given by the following result, which permits to find velocity profiles for the reference paths under the imposed constraints.

**Theorem 1:** A family of polynomials \( l_i(t-i), \ i \in \{0, 1, \ldots, N-1\} \) of degree \( q \geq 1 \) solves Problem 1 with resulting trajectory in the form (4), if and only if there exist symmetric and positive semidefinite matrices \( Q_i^j, R_i^j, Q_i^i, R_i^i \) such that the LMI problem (A1) is feasible. In this case, the coefficients of \( l_i(t-i) \) can be computed using (9) or (10).

An example related to the case of rotational motion with winding number \( N = 10, R = 3 \) and starting vertex on the elliptical barrier characterized by \( x_{d,0} = 1.6 \) is reported below. Solving the LMI problem (A1) it follows that \( q = 3 \) and the polynomials \( l_i(s) \) for \( i = \{0, \ldots, 9\} \) are given by (see figure 1):

\[
\begin{align*}
l_0(t) &= 0.26482 r^3 - 0.90733 r^2 + 1.64251 t^3, \\
l_1(t-1) &= 0.77602(t-1)^3 - 0.61631(t-1)^2 + 0.84569(t-1), \\
l_2(t-2) &= -0.47963(t-2)^3 + 0.45741(t-2)^2 + 1.02492(t-2), \\
l_3(t-3) &= 0.86908(t-3)^3 - 1.31563(t-3)^2 + 1.44652(t-3), \\
l_4(t-4) &= -0.22378(t-4)^3 + 0.70734(t-4)^2 + 0.51644(t-4), \\
l_5(t-5) &= 0.26473(t-5)^3 - 0.90733(t-5)^2 + 1.6426(t-5), \\
l_6(t-6) &= 0.77188(t-6)^3 - 0.61726(t-6)^2 + 0.84546(t-6), \\
l_7(t-7) &= -0.47873(t-7)^3 + 0.450305(t-7)^2 + 1.02568(t-7), \\
l_8(t-8) &= 0.8694(t-8)^3 - 1.3164(t-8)^2 + 1.447(t-8), \\
l_9(t-9) &= -0.22388(t-9)^3 + 0.70748(t-9)^2 + 0.5164(t-9).
\end{align*}
\]

(The LMI problem is solved by SeDuMi 1.1 (an open source LMI solvers for Matlab®) with Yalmip 3 as interface.)
The corresponding velocities \( \dot{x}_d(t) \) and \( \dot{y}_d(t) \) are shown in figure 2.

4. A tracking problem and its solution

In §3 a class of periodic trajectories in an elliptical billiards is found. The goal of this section is the design of a control law such that \( q(t) \) and its derivative asymptotically track the desired trajectory \( q_d(t) \) given in (4) and its derivative, respectively, where \( q(t) := [x(t), y(t)]^T \) is the position at time \( t \) of the controlled particle moving along the actual trajectory. Since many symbols will be used to state and solve the control problem topic of this section, table 1 reporting them with a brief description is added in order to help the reader.

Letting \( x_p(t) := [x(t), \dot{x}(t)]^T \), \( y_p(t) := [y(t), \dot{y}(t)]^T \), \( x_{pd}(t) := [x_d(t), \dot{x}_d(t)]^T \), \( y_{pd}(t) := [y_d(t), \dot{y}_d(t)]^T \), the tracking error at time \( t \) can be defined as

\[
\mathbf{e}_p(t) := \begin{bmatrix} x_p(t) - x_{pd}(t) \\ y_p(t) - y_{pd}(t) \end{bmatrix} \in \mathbb{R}^4.
\]

Notice that \( \mathbf{e}_p(t) \) has discontinuities not only at the impact times \( t_i \) but also at the integer times \( t_i^d = i \) (due to the jumps in the desired trajectory).

The presence of the Erdmann–Weierstrass corner conditions (2) and of the constraints (1c) and (1d) complicates the trajectory tracking problem as compared with the case of unconstrained mechanical systems. In Menini and Tornambè (2001), it is shown for a similar case that the error on the velocity
immediately after the impact times has in general absolute value greater than a given positive quantity. For this reason, the classical stability and attractivity properties are difficult (if not impossible) to be obtained and they have been properly amended in order to neglect in the analysis times belonging to infinitesimal intervals about the impact times, thus ensuring a sort of asymptotic stability for the error dynamics, similarly to what is proposed in Lakshmikantham et al. (1989) for impulsive differential systems. In the present work a controller based on the internal model principle (Francis and Wonham 1976, Joelianto and Williamson 1997) is considered. In particular, in order to obtain asymptotic tracking (in the sense specified in the subsequent Problem 2), in absence of impacts a continuous-time internal model of the desired trajectory is needed in the forward path of the feedback control system. The presence of such an internal model is guaranteed by a dynamic precompensator, whose state vector will be subject to discontinuities. Let $e_i(t)$ be the error between the actual state vector of the precompensator and its nominal value at time $t$, the control problem solved in this note can be stated as follows.

Problem 2: Find, if any, a piecewise continuous control law such that for each $\varepsilon > 0$, for each $t_0 \in \mathbb{R}^+$ and for each $\gamma \in (0, 1/2)$, there exists $\delta_{\varepsilon, \gamma} > 0$ such that if $|q(t_0) - Z_{\varepsilon, \gamma}q(t_0)| \in A$ and $||e_i^A(t_0), e_i^B(t_0)|| < \delta_{\varepsilon, \gamma}$, then the following properties hold for the closed-loop system:

1) $||e_i(t)|| < \varepsilon,$ \quad $\forall t \in \mathbb{R}^+, t > t_0, |t - \langle t \rangle| > \gamma,$

where $\langle t \rangle$ denotes the integer nearest to $t$. In the case in which $t$ is a half-integer, $\langle t \rangle$ denotes the smallest integer larger than $t$;

$$\lim_{i \to +\infty} ||e_i((i + \tau)^+)\| = 0, \quad \forall \tau \in (0, 1), \quad 2) \lim_{i \to +\infty} ||e_i((i + \tau)^-)\| = 0, \quad \forall \tau \in (0, 1),$$

where the limits are taken with $i$ being integer. (Actually, defining the position error vector as: $e_{pos}(t) := [x(t) - x_d(t), y(t) - y_d(t)]^T \in \mathbb{R}^2$, with slight modifications in the proof of the subsequent Theorem 2, the following stronger properties: 1bis: $\|e_{pos}(t)\| \leq \varepsilon, \forall t \geq t_0$ and 2bis: $\lim_{\tau \to +\infty} ||e_{pos}(t)\| = 0$ can be proved.)

4.1 The proposed control algorithm

The equations of motion of a particle moving in an elliptical billiards under the action of control forces have been obtained in §2. In absence of impacts (free-motion phase) the equations (1a) and (1b) can be rewritten as

$$\begin{align*}
x_p(t) &= A_p x_p(t) + B_p u_p(t), \quad \forall t \in (t_i, t_{i+1}), i \in \mathbb{Z}^+, \\
y_p(t) &= C_p x_p(t),
\end{align*}$$

(13a)

$$\begin{align*}
\dot{x}_p(t) &= A_p x_p(t) + B_p u_p(t), \quad \forall t \in (t_i, t_{i+1}), i \in \mathbb{Z}^+, \\
\dot{y}_p(t) &= C_p y_p(t)
\end{align*}$$

(13b)

where

$$A_p := \text{JordanBlock}(2, 0), \quad B_p := [0, 1]^T, \quad C_p := [1, 0].$$
As for the reference signals given in (4), they are polynomials in \( t - \lfloor t \rfloor \) of degree \( q \). Hence, for some \( \Omega_x, \Omega_y, v_x, v_y \), they can be written in the following form

\[
x_d(t) = \Omega_x e^{M(t - \lfloor t \rfloor)} v_x, \quad y_d(t) = \Omega_y e^{M(t - \lfloor t \rfloor)} v_y,
\]

where \( M = \text{JordanBlock}(q + 1, 0) \). In particular, defining \( \Omega_x^2 = \Omega_y^2 = [1, 0, \ldots, 0]^T \in \mathbb{R}^{q+1} \) one has

\[
\Omega_x e^{M(t - \lfloor t \rfloor)} = \Omega_y e^{M(t - \lfloor t \rfloor)} = \begin{bmatrix} 1, t - \lfloor t \rfloor, \frac{1}{2!}(t - \lfloor t \rfloor)^2, \\ \frac{1}{3!}(t - \lfloor t \rfloor)^3, \ldots, \frac{1}{q!}(t - \lfloor t \rfloor)^q \end{bmatrix},
\]

so that \( x_d(t) \) and \( y_d(t) \) can be obtained in the form (4),(6) by taking

\[
v_x(t) = \begin{bmatrix} x_{d,\lfloor t \rfloor} \\ a_1^{\lfloor t \rfloor} (x_{d,\lfloor t \rfloor} - x_{d,\lfloor t \rfloor}) \\ 2a_2^{\lfloor t \rfloor} (x_{d,\lfloor t \rfloor} - x_{d,\lfloor t \rfloor}) \\ \vdots \\ q! a_q^{\lfloor t \rfloor} (x_{d,\lfloor t \rfloor} - x_{d,\lfloor t \rfloor}) \end{bmatrix} \in \mathbb{R}^{q+1},
\]

\[
v_y(t) = \begin{bmatrix} y_{d,\lfloor t \rfloor} \\ a_1^{\lfloor t \rfloor} (y_{d,\lfloor t \rfloor} - y_{d,\lfloor t \rfloor}) \\ 2a_2^{\lfloor t \rfloor} (y_{d,\lfloor t \rfloor} - y_{d,\lfloor t \rfloor}) \\ \vdots \\ q! a_q^{\lfloor t \rfloor} (y_{d,\lfloor t \rfloor} - y_{d,\lfloor t \rfloor}) \end{bmatrix} \in \mathbb{R}^{q+1}.
\]

In order to solve the tracking problem, the internal model principle is considered, which possibly involves the design of a precompensator in the form

\[
\begin{cases}
\dot{x}_d(t) = A_x x_d(t) + B_x u_x(t), & \forall t \in (t_i, t_{i+1}), i \in \mathbb{Z}^+, \\
u_x(t) = C_x x_d(t), & \forall t \in (t_i, t_{i+1}), i \in \mathbb{Z}^+,
\end{cases}
\]

(15a)

\[
\begin{cases}
\dot{y}_d(t) = A_y y_d(t) + B_y u_y(t), & \forall t \in (t_i, t_{i+1}), i \in \mathbb{Z}^+, \\
u_y(t) = C_y y_d(t), & \forall t \in (t_i, t_{i+1}), i \in \mathbb{Z}^+.
\end{cases}
\]

(15b)

where \( x_d(t) \in \mathbb{R}^{q+1} \) and \( y_d(t) \in \mathbb{R}^{q+1} \) are the state vectors of the precompensator, \( u_x(t) \in \mathbb{R} \) and \( u_y(t) \in \mathbb{R} \) are the control inputs for the cascade precompensator+plant and the matrix \( A_x \) is chosen such that the dynamic matrix \( A \) in (18) for the cascade precompensator+plant has the same Jordan structure as \( M \). So, since the minimum polynomial of \( M \) is \( p(\lambda) = \lambda^{q+1} \), a realization \( (A_p, B_p, C_p) \) of the precompensator, with \( A_p \in \mathbb{R}^{(q-1) \times (q-1)} \), \( B_p \in \mathbb{R}^{q-1} \) and \( C_p \in \mathbb{R}^{q-1} \) is

\[
A_p = \text{JordanBlock}(q-1, 0),
\]

\[
B_p = [0 \ldots 0 \ 1]^T, \quad C_p = [1 \ 0 \ldots 0].
\]

Moreover, at any impact time \( t_i, i \in \mathbb{N} \), let the right-side values of the state vector of the controller be given by

\[
x_d(t_i^+) = A_x x_d(t_i),
\]

\[
y_d(t_i^+) = A_y y_d(t_i),
\]

(16a)

(16b)

where, in view of (14), \( A_{x_0}(t) := [0_{q-1,2}, L_{q-1}] y_{x}(t) = [2a_2 q_1 h_2(t), \ldots, q! a_q q_1 h_q(t)]^T \in \mathbb{R}^{q+1} \), and \( A_{y_0}(t) := [0_{q-1,2}, L_{q-1}] y_{y}(t) = [2a_2 q_1 w_2(t), \ldots, q! a_q q_1 w_q(t)]^T \in \mathbb{R}^{q+1} \) with \( h_1(t) := x_{d,i+1} - x_{d,i}, \quad w_1(t) := y_{d,i+1} - y_{d,i} \), and \( a_j, \ j = 2, \ldots, q \) being the coefficients of polynomials \( l_i(t) \) in (4). Notice that in (16) a discontinuity is imposed at each time \( t_i \) to the vector state of the precompensator, which is part of the proposed controller. In other words, at each impact time \( t_i \) system (15) is “stopped” and its state is reinitialized according to (16).

From (13) and (15) and defining the augmented state vector \( x(t) := [x_d(t), x_d(t)]^T \in \mathbb{R}^{2q+1} \) and \( y(t) := [y_d(t), y_d(t)]^T \in \mathbb{R}^{2q+1} \), one has

\[
\begin{cases}
\dot{x}_d(t) = A x_d(t) + B u_x(t), & \forall t \in (t_i, t_{i+1}), i \in \mathbb{Z}^+, \\
y_{d}(t) = C x_d(t), & \forall t \in (t_i, t_{i+1}), i \in \mathbb{Z}^+.
\end{cases}
\]

(17a)

\[
\begin{cases}
\dot{y}_d(t) = A y_d(t) + B u_y(t), & \forall t \in (t_i, t_{i+1}), i \in \mathbb{Z}^+, \\
y_{d}(t) = C y_d(t), & \forall t \in (t_i, t_{i+1}), i \in \mathbb{Z}^+.
\end{cases}
\]

(17b)

where the matrices \( A \in \mathbb{R}^{(q+1) \times (q+1)} \), \( B \in \mathbb{R}^{q+1} \) and \( C \in \mathbb{R}^{q+1} \) are given by

\[
A := \begin{bmatrix} A_p & B_p C_p \\ 0 & A_x \end{bmatrix}, \quad B := \begin{bmatrix} 0 \\ B_u \end{bmatrix}, \quad C := \begin{bmatrix} C_p & 0 \end{bmatrix}.
\]

(18)

Since the pair \((A, C)\) is observable and the eigenvalues of the dynamic matrix \( A \) are those of \( M \), then there exist unique initial conditions \( x_{d0}(t_i^+) \in \mathbb{R}^{q+1} \) and \( y_{d0}(t_i^+) \in \mathbb{R}^{q+1} \) such that

\[
\begin{cases}
\dot{x}_{d0}(t) = A x_{d0}(t), \\
x_{d0}(t) = C x_{d0}(t), & \forall t \in (t_i^+, t_{i+1}^+), i \in \mathbb{Z}, \\
x_{d0}(t_i^+) = v_x(t_i^+), & t_i^+ \in \mathbb{Z}, i \in \mathbb{Z},
\end{cases}
\]

(19a)

\[
\begin{cases}
\dot{y}_{d0}(t) = A y_{d0}(t), \\
y_{d0}(t) = C y_{d0}(t), & \forall t \in (t_i^+, t_{i+1}^+), i \in \mathbb{Z}, \\
y_{d0}(t_i^+) = v_y(t_i^+), & t_i^+ \in \mathbb{Z}, i \in \mathbb{Z},
\end{cases}
\]

(19b)
where \( \mathbf{x}_{\text{eff}}(t) \in \mathbb{R}^{q+1} \) and \( \mathbf{y}_{\text{eff}}(t) \in \mathbb{R}^{q+1} \) are defined for the desired trajectory according to the definition given in (17) for the controlled one, that is \( \mathbf{x}_{\text{eff}} := [x_{\text{eff}}^T, y_{\text{eff}}^T]^T \) and \( \mathbf{y}_{\text{eff}} := [y_{\text{eff}}^T, \mathbf{y}_{\text{eff}}^T]^T \). Moreover, \( \mathbf{v}_i := [x_{\text{eff}}^T, y_{\text{eff}}^T]^T \) and \( \mathbf{v}_i := [y_{\text{eff}}^T, \mathbf{y}_{\text{eff}}^T]^T \) for all \( i \in \mathbb{Z} \), with \( h_i(t) \) and \( w_i(t) \) being defined in (12) and \( a_j, j = 1, \ldots, q \) being the coefficients of polynomials \( l_i(t) \) in (4). At the impact times, defining

\[
\mathbf{z}(t) := \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \in \mathbb{R}^{2(q+1)}, \quad \mathbf{z}_d(t) := \begin{bmatrix} \mathbf{x}_{\text{eff}}(t) \\ \mathbf{y}_{\text{eff}}(t) \end{bmatrix} \in \mathbb{R}^{2(q+1)},
\]

the corner conditions (3) for the desired trajectory can be rewritten in terms of \( \mathbf{z}_d(t) \) as

\[
\mathbf{z}_d(t_i^+):= C(\theta_i^+)^T \mathbf{x}(t_i^+) + A_d(t_i^+), \quad i \in \mathbb{Z},
\]

whereas for the actual trajectory they become

\[
\mathbf{z}(t_i^-):= C(\theta_i^-)^T \mathbf{x}(t_i^-) + A_d(t_i^-), \quad i \in \mathbb{N},
\]

where \( \theta_i^+ \in \mathbb{R} \) and \( \theta_i \in \mathbb{R} \) are the polar angles relative to the impacts occurring at the times \( t_i^+ \) and \( t_i \) for the desired and actual trajectory, respectively, \( A_d(i) := [0, 0, A_{\text{eff}}^T(i), 0, 0, A_{\text{eff}}^T(i)]^T \in \mathbb{R}^{2(q+1)} \) and \( A \) is the transition matrix from state to another. Essentially, those are the events listed above with the threshold \( \sigma \) defined as the maximum allowed time interval with the control switched off. This small positive number \( \sigma \) guarantees to avoid to keep down the control for too much time, that is bad for the tracking purpose.

Assuming that the plant state \( \mathbf{z}(t) := [x_i^T(t), y_i^T(t)]^T \in \mathbb{R}^{2(q+1)} \) is available, during the interval of time with the control active, the control law is

\[
\begin{align}
\mathbf{u}_{x_e} &:= \mathbf{K}_x (\mathbf{x}_e - \mathbf{x}_{\text{eff}}), \\
\mathbf{u}_{y_e} &:= \mathbf{K}_y (\mathbf{y}_e - \mathbf{y}_{\text{eff}}),
\end{align}
\]

where \( \mathbf{u}_{x_e} \in \mathbb{R} \) and \( \mathbf{u}_{y_e} \in \mathbb{R} \) are the control inputs in (15) and \( \mathbf{K}_x^T \in \mathbb{R}^{(q+1) \times (q+1)} \) and \( \mathbf{K}_y^T \in \mathbb{R}^{(q+1) \times (q+1)} \) are chosen such that all eigenvalues of \( \mathbf{A} + \mathbf{B} \mathbf{K}_x \) and \( \mathbf{A} + \mathbf{B} \mathbf{K}_y \) have real part less than or equal to \( -\eta \), with \( \eta \) being a positive constant and \( \mathbf{A}, \mathbf{B} \) are given in (18). During the intervals of time with the control off, one has \( u_{x_e} = 0 \) and \( u_{y_e} = 0 \).

**Remark 2**: The aim to switch off of the control is to improve the behavior of the considered system. Some reasons and possible scenarios can be considered in order to justify such a choice. As a matter of fact, assume that the control is always active and that the nominal trajectory impacts before the corresponding impact of the controlled trajectory, the latter starts to “change” in order to track the new reference, that is bouncing away from the boundary; hence, even if the actual trajectory hits the boundary, such an impact can be very far from its nominal value; in such conditions it is quite easy that the actual trajectory does not impact at all. Still with the control always active, if the controlled trajectory impacts before the nominal one, one can easily have an accumulation...
point of the impact times of the actual trajectory close to the impact of the desired trajectory; although this fact is not necessarily bad for the tracking objective, it seems better to avoid as much as possible such a possibility. Through many simulations, it has been shown that the proposed algorithm permits to obtain the tracking purpose also when the initial conditions are quite far from the desired ones (see figure 4). Moreover, also when the actual and the nominal trajectories are very close, it is observed that switching off the control between the impacts improves the tracking.

4.2 Formal result

Assumption 1: If the initial time \( t_0 \) is an impact time for the desired trajectory (i.e., \( t_0 \in \mathbb{Z}^+ \)), then \( \mathbf{z}_d(t_0) = \mathbf{z}_d(t_0^+) \), i.e., \( \mathbf{z}_d(t_0) = [v_x(t_0), v_y(t_0)]^T \), where \( v_x(\cdot) \) and \( v_y(\cdot) \) are defined in (14).

By choosing the initial conditions sufficiently close to the desired ones (as allowed by the stability-like requirements in Problem 2) and under the Assumption 1 (Assumption 1 with the hypothesis of the Problem 2 permits to avoid that the actual trajectory hits the boundary just after \( t_0 \), for any integer \( i > [t_0] \)
the impact time $t_i \in \mathbb{R}^+$ of the actual trajectory can be forced to be close to the impact time $i$ of the desired trajectory (see proof of the subsequent Theorem 2), so that the proposed control algorithm coincides with

$$u_{x_c}(t) = \begin{cases} K_x(x_c(t) - x_{ed}(t)), & \forall t \in (t^M_i, t^{m}_i), i \in \mathbb{Z}, i \geq \lfloor t_0 \rfloor, \\ 0, & \text{otherwise}, \end{cases}$$

$$u_{y_c}(t) = \begin{cases} K_y(y_c(t) - y_{ed}(t)), & \forall t \in (t^M_i, t^{m}_i), i \in \mathbb{Z}, i \geq \lfloor t_0 \rfloor, \\ 0, & \text{otherwise}, \end{cases}$$

(23a)

$$\text{(23b)}$$

where $t^M_{\lfloor t_0 \rfloor} := t_0$ and, for any $t^{M}_i := t_0, t^{m}_i, i \in \mathbb{Z}, i \geq \lfloor t_0 \rfloor + 1,$ one defines $t^{m}_i := \min(t_i, t^{M}_i)$ and $t^M_i := \max(t_i, t^{M}_i)$.

**Theorem 2:** Under the Assumption 1, there exists $\eta^* \in \mathbb{R}^+$ such that the controller (15),(16) and (23) solves Problem 2, for any $\eta \geq \eta^*$.

**Proof:** In the following, considering the extended tracking error vector defined as

$$e(t) := z(t) - z_d(t),$$

(24)

it is shown how the properties 1) and 2) of Problem 2 can be guaranteed also when $e(t) = e_i(t)$, thus satisfying a stronger requirement. The proof can be carried out by means of the steps described below, based on the following facts whose proofs are given in Appendix B.

$$\exists \delta_1, M_2, M_3 \in \mathbb{R}^+ : \|e(t^{m}_i)\| < \delta_1, \left\| \frac{\Delta t_i}{\Delta \theta_i} \right\| < \delta_1$$

$$\Rightarrow \|e(t^{M}_i)\| \leq M_2\|e(t^{m}_i)\| + M_3 \left\| \frac{\Delta t_i}{\Delta \theta_i} \right\|,$$

(26)

$$\forall \eta \in \mathbb{R}^+, \|e(t)\| \leq L(\eta)e^{-\gamma(t-t^0)}\|e(t^{M}_i)\|, \forall t \in (t^M_i, t^{m}_{i+1}),$$

(27a)

$$\forall \delta^* > 0, \forall T > 0, \exists \eta^* > 0 : \eta > \eta^* \Rightarrow L(\eta)e^{-\gamma T} < \delta^*,$$

(27b)

where $\Delta t_i := t_i - t^{M}_i, \Delta \theta_i := \theta_i - \theta^{M}_i$ and, in view of the $N$-periodicity of the desired trajectory, (25) and (26) hold for any $t_i \in \mathbb{Z}, i \geq \lfloor t_0 \rfloor + 1$ and (27) holds for any $t_i \in \mathbb{Z}, i \geq \lfloor t_0 \rfloor$.

**Remark 3:** Roughly speaking, (25) and (26) show that the minimal amount of continuity needed to obtain the stability properties is still present despite the jumps affecting the system. The fact that such properties can be proved is essential as this represents one technical difficulty that is intrinsic to non-smooth systems.

**Step (i):** In view of (25) and (26) and in order to guarantee that for any $t_i \in \mathbb{Z}, i \geq \lfloor t_0 \rfloor + 1$ the actual impact time $t_i$ belongs to the interval $(t^M_i - \gamma, t^{m}_i + \gamma)$ for a fixed $\gamma \in (0, 1/2)$, one can take $\|e(t^{m}_i)\| < \delta_0$ where $\delta_0 := \min(\delta_0, \delta_1/ M_1, \gamma/ M_1)$.

**Step (ii):** Putting together (25) and (26), the following inequality is obtained

$$\|e(t^{M}_i)\| \leq (M_2 + M_3 M_1)\|e(t^{m}_i)\| =: \beta\|e(t^{m}_i)\|,$$

(28)

where $\beta := M_2 + M_3 M_1$.
Step (iii): Using (27a) and (28), for any $i \in \mathbb{Z}$, $i \geq [t_0] + 1$ one has (by Step (i), the minimum flight-time in free motion is $1 - 2\gamma$)

$$\|e(t_i^-)\| \leq L(\eta)e^{-\eta(t_i^- - t_i^+)}\|e(t_i^+)\|
\leq L(\eta)e^{-\eta(1-2\gamma)}\|e(t_i^-)\|.$$  

At this point, taking in (28) $\varepsilon^* \leq \xi/\beta$ with $0 \leq \xi < 1$, there exists $\eta^*$ such that, for any $\eta > \eta^*$, it follows that $Le^{-\eta(1-2\gamma)} \leq \xi$, which implies (from now on, $\eta$ is fixed and hence $L$ is a real positive constant (dependence on $\eta$ is omitted))

$$\|e(t_i^-)\| \leq \xi\|e(t_i^-)\|, \quad \xi \in (0, 1),$$  

for any $i \in \mathbb{Z}, i \geq [t_0] + 1$.

Step (iv): For a generic time interval $(t_i^M, t_{i+1}^-)$, $i \in \mathbb{Z}, i \geq [t_0] + 1$, from (27a) one has

$$\|e(t)\| \leq Le^{-\eta(t-t_i^-)}\|e(t_i^M)\| \leq L\beta\|e(t_i^-)\|,$$

and taking $\|e(t_i^-)\| < \frac{\tilde{\varepsilon}}{\tilde{\beta}}$ with $\tilde{\beta} := \min(\tilde{\beta}_0, \varepsilon/(L\beta))$ for a generic $\varepsilon > 0$ it follows that $\|e(t)\| < \varepsilon$ for any $i \in (t_i^M, t_{i+1}^-), i \in \mathbb{Z}, i \geq [t_0] + 1$. From (29), it is clear that if $\|e(t_i^-)\| < \tilde{\beta}_0$, then $\|e(t_{i+1}^-)\| < \tilde{\beta}_0$, in fact

$$\|e(t_{i+1}^-)\| \leq \xi\|e(t_i^-)\| < \xi\tilde{\beta}_0 < \frac{\tilde{\varepsilon}}{\tilde{\beta}}.$$  

Now, in order to guarantee that $\|e(t_i^-)\| < \frac{\tilde{\varepsilon}}{\tilde{\beta}}$ for any $i \in \mathbb{Z}, i \geq [t_0] + 1$ and $\|e(t)\| < \varepsilon$ for any $t \in (t_i^M, t_{i+1}^-)$, it is sufficient to take $\delta_{k, \gamma} := \min(\delta_{k_0}/L, \varepsilon/L)$. In summary, for any $i \in \mathbb{Z}, i \geq [t_0]$ the following result has been obtained

$$\|e(t_i^-)\| < \delta_{k, \gamma} \Rightarrow \|e(t)\| < \varepsilon, \quad \forall t \in (t_i^M, t_{i+1}^-),$$  

and since $|\Delta t_i| < \gamma, i \in \mathbb{Z}, i \geq [t_0] + 1$ (see Step (i)), then (30) implies property 1 in Problem 2.

To complete the proof it remains to prove property 2 in Problem 2.

Step (v): If $\|e(t_i^-)\| < \delta_{k, \gamma}$, then applying iteratively (29) one obtains

$$\|e(t_i^-)\| < \xi\|e(t_{i+1}^-)\| < \xi^2\|e(t_{i+2}^-)\| < \xi^i\|e(t_{i+1}^-)\| < \xi^i\|e(t_i^-)\| \rightarrow 0$$  

since $\xi \leq (0, 1).$

From (31) and (25) it follows that

$$\|e(t_i^-)\| \rightarrow 0 \Rightarrow \{ \begin{array}{l}
  t_i \rightarrow t_i^M \\
  \theta_i \rightarrow \theta_i^M
\end{array} \quad \text{as } i \rightarrow +\infty.$$  

Moreover, with similar reasonings it is easy to see that in any interval $(t_i^M, t_{i+1}^-)$

$$\|e(t)\| \rightarrow 0 \quad \text{as } i \rightarrow \infty,$$

and by this result and (32) it follows that

$$\forall \tau \in (0, 1), \exists \theta^*: i > i^* \Rightarrow t_i^M + \tau \in (t_i^M, t_{i+1}^-) \quad \text{and } \quad e(t_i^M + \tau) \rightarrow 0 \quad \text{as } i \rightarrow +\infty.$$  

Since $t_i \rightarrow t_i^M$ as $i \rightarrow +\infty$, time $t_i^M + \tau, \forall \tau \in (0, 1)$ is an impact time neither for the actual trajectory nor for the desired one. Therefore, $\lim_{i \rightarrow +\infty} \|e((t_i^M + \tau)^+)\| = \lim_{i \rightarrow +\infty} \|e(t_i^M + \tau)\| = 0$. This last fact, together with the definition of $\varepsilon(t)$ given in (24), proves property 2 of Problem 2. □

5. Simulation examples

In this last section two simulation examples are given in order to show the effectiveness of the proposed control law. Considering an elliptical barrier centered at the origin and characterized by $a = 4$ and $b = 2$, first the case of rotational motion with winding number $(N = 7, R = 2)$ and starting vertex $v_0 = [0.3, 1.9944]$ being on the elliptical boundary is taken into account, then the librational motion with winding number $(N = 4, R = 1)$ and starting vertex $v_0 = [1.3, 1.8914]$ being on the elliptical boundary is considered. In both cases the velocity profiles solving Problem 1 are obtained by Theorem 1. Finally, choosing $K_x$ and $K_p$ in (22) such that all the eigenvalues of the closed-loop system are moved at $\eta = 7$ and setting the value of the threshold $\sigma$ to 0.15, the behaviour of the controlled trajectories during the first 9.5 seconds of motion, starting from zero initial conditions at the initial time $t_0 = 0.5$, can be observed in figures 5 and 6 for the rotational case and in figures 7 and 8 for the librational case. Though the switching algorithm with the threshold $\sigma$ has been used, due to the fact that the desired trajectory is periodic and the system to be controlled is time-invariant, in order to obtain simulation examples of the situation considered in Theorem 2 (when it is assumed that the initial error is sufficiently small so that the threshold $\sigma$ is never active) one can consider the same examples proposed here starting from a time $t_0^* \geq t_0$, such that the controlled trajectory at time $t_0^*$ is close to the nominal one (e.g., for both examples depicted here one can take (roughly) $t_0^* = 3$). The simulation was performed in Matlab©.
Figure 5. Rotational motion \((N = 7, R = 2)\): the inner caustic curve (dotted) with the desired (dashed) trajectory, which is completely overlapped with the actual (solid) one, in the \(xy\)-plane (a) and time behaviour of the desired (dashed) and actual (solid) positions (b) and velocities (c): (a) \(q(t), q_d(t)\); (b) \(x(t), x_d(t)\) and \(y(t), y_d(t)\); (c) \(\dot{x}(t), \dot{x}_d(t)\) and \(\dot{y}(t), \dot{y}_d(t)\).
using the event option of the ODE solvers for impact detection and handling.

6. Conclusion

After giving a procedure for determining admissible periodic trajectories inside an elliptical billiards, a suitable tracking control problem has been formulated and solved using a dynamic compensator, whose state is subject to discontinuities and whose structure is based on the internal model principle. In particular, the proposed results constitute a first step towards the definition of a non-smooth version of the internal model principle, valid for a class of dynamical systems whose state is subject to jumps. In the present paper all the results are obtained under the assumption that the coefficient of restitution \( e \) is equal to 1. If \( e < 1 \), different desired paths should be computed in order to take into account that, at each impact, the angle of incidence is different from the angle of reflection. In such a case one first possibility, for sufficiently large \( e \), still requiring that the path between two consecutive impacts is a straight line, would be to compute a closed desired path numerically by guessing an initial segment and adjusting it in order to obtain a closed orbit. Otherwise, and such a second possibility would be feasible for all values of \( e \), one should accept that the path between two consecutive impacts is not a straight line. On the other hand, keeping the desired
Figure 7. Librational motion ($N = 4, R = 1$): the inner caustic curve (dotted) with the desired (dashed) trajectory, which is completely overlapped with the actual (solid) one, in the $xy$-plane (a) and time behavior of the desired (dashed) and actual (solid) positions (b) and velocities (c): (a) $q(t)$, $q_d(t)$; (b) $x(t)$, $x_d(t)$ and $y(t)$, $y_d(t)$; (c) $\dot{x}(t)$, $\dot{x}_d(t)$ and $\dot{y}(t)$, $\dot{y}_d(t)$. 
Trajectories proposed in this paper, if \( e < 1 \), only impulsive control can be a solution of Problem 2: as a matter of fact, in order for \( (q(t), \dot{q}(t)) = (q_0(t), \dot{q}_0(t)) \) to be a piecewise solution of the closed-loop system for \( t \geq t_0 \), it is necessary that the kinetic energy lost at each impact time due to the coefficient of restitution less than one is restored so that the kinetic energy immediately after the impact takes its value immediately before the impact, through the action of the control: this is impossible as the control forces are not impulsive.

Although in this paper it has been assumed that the impacts can be detected in real time a slightly modified control problem involving a relaxed tracking requirement can be solved even without such an assumption.

**Appendix A: The LMI problem to compute velocity profiles for the reference trajectories**

From Lemma 1, the conditions (8) are satisfied if and only if there exist symmetric and positive semidefinite matrices \( Q_i^1, R_i^3, Q_i^f \) and \( R_i^2 \) for \( i = 0, \ldots, N - 1 \) such that

\[
a_i q_{i-1} s^{q_i - 1} + \cdots + a_0 s + a_i s + a_0 \geq 0 \quad \Leftrightarrow \quad (9),
\]

and

\[
1 - a_i q_{i-1} s^{q_i - 1} - \cdots - a_0 s - a_i s + a_0 \geq 0 \quad \Leftrightarrow \quad (10),
\]
where the Hankel matrices $H_i$ are given by

$$H_0 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad H_1 = \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},$$
$$H_2 = \begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}, \quad \ldots,
$$

for $i \in \{0, 1, \ldots, 2m\}$, whereas $H_i = 0$ for $i < 0$ or $i > 2m$ with $m = q/2$ ($q$ even) or $m = (q - 1)/2$ ($q$ odd).

In order to guarantee (5), for any integer $i = 0, 1, \ldots, N - 1$ the following relations in terms of the decision variables are obtained:

$$l_i(0) = 0 \iff \sum_{q} \left( \frac{\text{tr}(Q_i^1 H_0)}{\gamma_q} + \frac{\text{tr}(Q_i^1 H_0) + \text{tr}(R_i^1 (H_1 - H_0))}{\gamma_q} + \cdots \right) = 0,$$

$$l_i(1) \equiv \frac{\text{tr}(Q_i^1 H_0) + \text{tr}(Q_i^1 H_0) + \text{tr}(R_i^1 (H_1 - H_0))}{\gamma_q} + \cdots = 1.$$

Since the coefficients $\gamma_0, \gamma_1, \ldots, \gamma_q$ in (9) and (10) are relative to the same polynomial $l_i(\cdot)$, one has

$$\begin{align*}
\text{tr}(R_i^1 H_0) &= 1 + \text{tr}(R_i^2 H_0), \\
\text{tr}(Q_i^1 H_0) + \text{tr}(R_i^1 (H_1 - H_0)) &= -(\text{tr}(Q_i^1 H_0) + \text{tr}(R_i^2 (H_1 - H_0))), \\
\vdots \quad \vdots
\end{align*}$$

Finally, taking into account the corner conditions given by (7), for any $i = 0, \ldots, N - 1$, one obtains

$$\begin{align*}
\frac{\text{tr}(Q_i^1 H_0) + \text{tr}(R_i^1 (H_1 - H_0))}{\gamma_q} (x_{d,i+1} - x_{d,i}) &= \left( q \frac{\text{tr}(Q_i^1 H_{q-1}) + \text{tr}(R_i^1 (H_q - H_{q-1}))}{\gamma_q} \right) \\
&+ (q - 1) \frac{\text{tr}(Q_i^1 H_{q-2}) + \text{tr}(R_i^1 (H_{q-1} - H_{q-2}))}{\gamma_q} \\
&\times (C_1(q_d(i)) (x_{d,i} - x_{d,i-1}) + C_2(q_d(i)) (y_{d,i} - y_{d,i-1})),
\end{align*}$$

and

$$\begin{align*}
\frac{\text{tr}(Q_i^1 H_0) + \text{tr}(R_i^1 (H_1 - H_0))}{\gamma_q} (y_{d,i+1} - y_{d,i}) &= \left( q \frac{\text{tr}(Q_i^1 H_{q-1}) + \text{tr}(R_i^1 (H_q - H_{q-1}))}{\gamma_q} \right) \\
&+ (q - 1) \frac{\text{tr}(Q_i^1 H_{q-2}) + \text{tr}(R_i^1 (H_{q-1} - H_{q-2}))}{\gamma_q} \\
&\times (C_2(q_d(i)) (x_{d,i} - x_{d,i-1}) - C_1(q_d(i)) (y_{d,i} - y_{d,i-1})),
\end{align*}$$

where $C_1(\cdot)$ and $C_2(\cdot)$ are defined in (3) and $[x_{d,i}, y_{d,i}]^T$ represents the position of the $i$th vertex of the reference trajectory.

In summary, collecting the results above and defining

$$\begin{align*}
\alpha_0 &:= \text{tr}(R_i^1 H_0), \\
\alpha_1 &:= \text{tr}(Q_i^1 H_0) + \text{tr}(R_i^1 (H_1 - H_0)), \\
&\vdots \\
\alpha_q &:= \text{tr}(Q_i^1 H_{q-1}) + \text{tr}(R_i^1 (H_{q} - H_{q-1})), \\
\bar{a}_0 &:= 1 - \text{tr}(R_i^2 H_0), \\
\bar{a}_1 &:= -(\text{tr}(Q_i^2 H_0) + \text{tr}(Q_i^2 (H_1 - H_0))), \\
&\vdots \\
\bar{a}_q &:= -(\text{tr}(Q_i^2 H_{q-1}) + \text{tr}(Q_i^2 (H_{q} - H_{q-1}))),
\end{align*}$$

and

$$\begin{align*}
h_1(i) &:= x_{d,i+1} - x_{d,i}, \\
h_2(i) &:= C_1(q_d(i)) (x_{d,i} - x_{d,i-1}) + C_2(q_d(i)) (y_{d,i} - y_{d,i-1}), \\
w_1(i) &:= y_{d,i+1} - y_{d,i}, \\
w_2(i) &:= C_2(q_d(i)) (x_{d,i} - x_{d,i-1}) - C_1(q_d(i)) (y_{d,i} - y_{d,i-1}),
\end{align*}$$

the following LMI problem to solve in the $4N$ symmetric and positive semidefinite matrices $Q_i^1, R_i^1, Q_i^2$ and $R_i^2$.
for $i = 0, 1, \ldots, N - 1$ is obtained

$$
\begin{align*}
N \text{ equalities:} & \\
& d_0^0 = 0, \\
& d_0^{N-1} = 0, \\
& d_q^0 + d_q^1 + \cdots + d_q^N = 1, \\
N \text{ equalities:} & \\
\begin{align*}
& a_q^0 h_1(0) = (q a_q^{N-2} + (q - 1) a_q^{N-1}), \\
& + \cdots + a_q^{N-1}) h_2(0), \\
& a_q^0 w_1(0) = (q a_q^{N-2} + (q - 1) a_q^{N-1}), \\
& + \cdots + a_q^{N-1}) w_2(0), \\
2N \text{ equalities:} & \\
& a_q^{N-1} h_1(N - 1) = (q a_q^{N-2} + (q - 1) a_q^{N-1}), \\
& + \cdots + a_q^{N-1}) h_2(N - 1), \\
& a_q^{N-1} w_1(N - 1) = (q a_q^{N-2} + (q - 1) a_q^{N-1}), \\
& + \cdots + a_q^{N-1}) w_2(N - 1), \\
(q + 1)N \text{ equalities:} & \\
& a_q^{N-1} = \tilde{a}_q^{N-1}, \\
& \vdots \\
& a_q^0 = \tilde{a}_q^0, \\
4N \text{ inequalities:} & \\
& Q_1^i \geq 0, R_1^i \geq 0, Q_2^i \geq 0, R_2^i \geq 0.
\end{align*}
\end{align*}
$$

\begin{align}
\text{Remark B1:} \quad \text{Since the desired trajectory is } N\text{-periodic, all the results obtained in the following considering } i \in \{[t_0] + 1, \ldots, N + [t_0]\} = I, \text{ remain proved for } i \in \mathbb{Z}, i \geq [t_0] + 1. \\
\end{align}

Define, for any $i \in I,$

$$
g_i(x_1, y_1, \Delta \theta) := \begin{bmatrix} g_{1}(x_1, y_1) \\
g_{2,i}(x_1, y_1, \Delta \theta) \end{bmatrix} : \mathbb{R}^3 \to \mathbb{R}^2,
$$

where $g_1: \mathbb{R}^2 \to \mathbb{R}$ and $g_{2,i}: \mathbb{R}^3 \to \mathbb{R}$ are given by (the dependence of $g_{2,i}$ upon $i$ is due to the fact that the straight-line defined by such an implicit function changes with $\theta_i$, whereas for all the vertices of the actual trajectory the implicit function $g_1$ remains unchanged)

$$
g_{1}(x_1, y_1) := \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1, \quad (B1a)
$$

$$
g_{2,i}(x_1, y_1, \Delta \theta) := \sin(\theta_i + \Delta \theta)x_1 - \cos(\theta_i + \Delta \theta)y_1. \quad (B1b)
$$

Therefore, from (B1a) and (B1b) it follows that $g_i(x_1, y_1, \Delta \theta) = 0,$ for any $i \in I,$ at the intersection between the elliptical boundary and a straight-line starting from the origin with direction normal to the vector $[\sin(\theta_i^d + \Delta \theta), -\cos(\theta_i^d + \Delta \theta)]^T.$

If $x_1$ and $y_1$ are defined as the position coordinates for the actual trajectory at the impact time $t_i, i \in I,$ that is $x_1 := x(t_i)$ and $y_1 := y(t_i)$, then they can be written as

$$
\begin{bmatrix} x_1 \\
y_1 \end{bmatrix} = \begin{bmatrix} h_{i,1}(\Delta t_i, e(t_i^{-})) \\
h_{i,2}(\Delta t_i, e(t_i^{-})) \end{bmatrix}, \quad (B2)
$$

where $h_{i,1,i,2,3}: \mathbb{R}^{2(q+1)+1} \to \mathbb{R}$ and $\Delta t_i := t_i - t_i^d$.

\begin{align}
\text{Proof of (B2):} \quad \text{In the following, in order to show that } x_1 \text{ and } y_1 \text{ can be expressed as functions of } \Delta t_i \text{ and } e(t_i^{-}) \text{ for any } i \in I, \text{ the two possible cases are considered separately.}
\end{align}

Case a) $t_i^m = t_i, \quad t_i^M = t_i^d,$ so that $\Delta t_i < 0$ and $z(t_i^d) = z(t_i^{-}).$ In this case, one has

$$
\begin{align}
z(t_i) = z(t_i^{-}) = e(t_i^{-}) + z_i(t_i^{-}) = e(t_i^{-}) \\
+ e^{-\lambda_i(t_i^{-})}z_i(t_i^{-}) = e(t_i^{-}) + e^{\lambda_i(t_i^{-})}z_i(t_i^{-}) \\
\end{align}
$$

Case b) $t_i^m = t_i^d, \quad t_i^M = t_i,$ so that $\Delta t_i > 0$ and $z(t_i^d) = z(t_i^{-}),$ where $z(t)$ is defined in (20). In this
case, one has

\[
x(t_i^-) = x(t_i^-) = e^{\tilde{A}(t_i^- - t_i)}x(t_i^-) = e^{\tilde{A}(t_i^- - t_i)}z(t_i^-) \\
= e^{\tilde{A} \Delta t} x(t_i^-) + e^{\tilde{A} \Delta t} z_x(t_i^-).
\]

From (B3) and (B4), for any \( i \in I_N \), it is easy to see that

\( x(t_i^-) \) is a function of \( \Delta t_i \) and \( e(t_i^-) \) and given

that \( x(t_i^-) = x(t_i) \) and \( y(t_i^-) = y(t_i) \) are the first and the

((\( q + 1 \)) + 1)-th component of \( z(t_i^-) \), respectively, there

exist two functions \( h_1,i \) and \( h_2,i \) of \( \Delta t_i \) and \( e(t_i^-) \) such

that (B2) holds.

Moreover, \( z(t_i^-) \) is an analytic function of \( \Delta t_i \) and \( e(t_i^-) \)

for \( \Delta t_i < 0 \) and \( \Delta t_i > 0 \), for any \( i \in I_N \). In the hyper-

plane characterized by \( \Delta t_i = 0 \), it is easy to see that

\( z(t_i^-) \) is a continuous function with respect to its

variables. As a matter of fact, one can observe that in

a whole period (i.e., for any \( i \in I_N \))

Case a)

\[
z(t_i^-)_{|_{\Delta t_i = 0}} = e(t_i^-) + I z_x(t_i^-) = e(t_i^-) + z_x(t_i^-);
\]

Case b)

\[
z(t_i^-)_{|_{\Delta t_i = 0}} = e(t_i^-) + I z_x(t_i^-) = e(t_i^-) + z_x(t_i^-),
\]

and the continuity of \( z(t_i^-) \) in \( \Delta t_i = 0 \) implies the continuity

also for \( x(t_i), y(t_i), x(t_i^-) \) and \( y(t_i^-) \) with respect to

\( \Delta t_i \), since they are simply four components of \( z(t_i^-) \).

In other terms, defining \( V_i^T \in \mathbb{R}^{2(q+1)}, V_1^T \in \mathbb{R}^{2(q+1)}, 

V_2^T \in \mathbb{R}^{2(q+1)} \) and \( V_3^T \in \mathbb{R}^{2(q+1)} \) as

\[
V_1 := [1, 0, 0, \ldots, 0], \quad V_2 := [0, 0, 0, 1, 0, \ldots, 0], 

V_3 := [0, 1, 0, \ldots, 0], \quad V_4 := [0, 0, 0, 1, 0, \ldots, 0],
\]

and since

\[
e(t_i^-) + z_x(t_i^-) = z(t_i^-),
\]

in both cases a) and b) one has (take the derivative with

respect to \( \Delta t \) or \( t \) is equivalent, that is \( \partial x(\cdot)/\partial \Delta t = \dot{x}(\cdot), 

\]

since \( \Delta t := t - i, i \in \mathbb{Z} \)

\[
x(t_i^-)_{|_{\Delta t_i = 0}} = V_1 z(t_i^-), \\
y(t_i^-)_{|_{\Delta t_i = 0}} = V_2 z(t_i^-), \\
\dot{x}(t_i^-)_{|_{\Delta t_i = 0}} = V_3 z(t_i^-), \\
\dot{y}(t_i^-)_{|_{\Delta t_i = 0}} = V_4 z(t_i^-).
\]

Moreover, regarding the partial derivatives, after some

tedious computations, in both cases a) and b) the following

results hold

\[
\frac{\partial z(t_i^-)}{\partial e(t_i^-)_{|_{\Delta t_i = 0}}} = I = \left\{ \begin{array}{c}
\frac{\partial V_1 z(t_i^-)}{\partial e(t_i^-)_{|_{\Delta t_i = 0}}} \quad \frac{\partial x(t_i)}{\partial e(t_i^-)_{|_{\Delta t_i = 0}}} \\
\frac{\partial V_2 z(t_i^-)}{\partial e(t_i^-)_{|_{\Delta t_i = 0}}} \quad \frac{\partial y(t_i)}{\partial e(t_i^-)_{|_{\Delta t_i = 0}}} \end{array} \right\} = V_1,
\]

\[
\frac{\partial z(t_i^-)}{\partial e(t_i^-)_{|_{\Delta t_i = 0}}} = I = \left\{ \begin{array}{c}
\frac{\partial V_1 z(t_i^-)}{\partial e(t_i^-)_{|_{\Delta t_i = 0}}} \quad \frac{\partial x(t_i)}{\partial e(t_i^-)_{|_{\Delta t_i = 0}}} \\
\frac{\partial V_2 z(t_i^-)}{\partial e(t_i^-)_{|_{\Delta t_i = 0}}} \quad \frac{\partial y(t_i)}{\partial e(t_i^-)_{|_{\Delta t_i = 0}}} \end{array} \right\} = V_2.
\]
and

\[
\frac{\partial z(t_i)}{\partial \Delta \theta_i} \bigg|_{\Delta t_i=0} = 0 \Rightarrow \begin{cases} \frac{\partial V_1(z(t_i))}{\partial \Delta \theta_i} \bigg|_{\Delta t_i=0} = \frac{\partial x(t_i)}{\partial \Delta \theta_i} \bigg|_{\Delta t_i=0} = 0 \\ \frac{\partial V_2(z(t_i))}{\partial \Delta \theta_i} \bigg|_{\Delta t_i=0} = \frac{\partial y(t_i)}{\partial \Delta \theta_i} \bigg|_{\Delta t_i=0} = 0 \end{cases}.
\]

Therefore, it is clear that, for any \( i \in I_N \), the functions \( g_i \) are continuously differentiable also on the hyperplane \( \Delta t_i = 0 \), in fact from the results above it follows that the partial derivatives of \( g_1 \) and \( g_{2,i} \) with respect to \( \Delta t_i, \Delta \theta_i \) and \( e(t_i^-) \) are in both cases exactly the same when \( \Delta t_i = 0 \).

At this point, using (B2) the functions \( \hat{g}_i : \mathbb{R}^{2(q+1)+2} \to \mathbb{R}^2 \) can be defined, for any \( i \in I_N \), as

\[
\hat{g}_i(\Delta t_i, \Delta \theta_i, e(t_i^-)) := g_i(h_1, (\Delta t_i, e(t_i^-)), \quad h_2, (\Delta t_i, e(t_i^-)), \Delta \theta_i),
\]

where

\[
h_1, (\Delta t_i, e(t_i^-)) := \begin{cases} V_1(e(t_i^-) + e^{\Delta t_i}z_a(t_i^-)), & \Delta t_i \leq 0 \\ V_1 e^{\Delta t_i}(e(t_i^-) + z_a(t_i^-)), & \Delta t_i \geq 0 \end{cases},
\]

and

\[
h_2, (\Delta t_i, e(t_i^-)) := \begin{cases} V_2(e(t_i^-) + e^{\Delta t_i}z_a(t_i^-)), & \Delta t_i \leq 0 \\ V_2 e^{\Delta t_i}(e(t_i^-) + z_a(t_i^-)), & \Delta t_i \geq 0 \end{cases}.
\]

Remark B2: In view of the results above, there exists a neighbourhood \( \Omega \) of \( \Delta t_i = 0, \Delta \theta_i = 0 \) and \( e(t_i^-) = 0 \) such that (B5) are continuously differentiable on that neighbourhood for any \( i \in I_N \).

Lemma B.1 (implicit function theorem, see, e.g., Apostol (1974)): Let \( \tilde{g}_i(\chi, \psi) : \mathbb{R}^{2+3(q+1)} \supseteq \Omega \to \mathbb{R}^2 \) be continuously differentiable on the open set \( \Omega \). Let \( (\chi^0, \psi^0) \) be a point in \( \Omega \) for which \( \tilde{g}_i(\chi^0, \psi^0) = 0 \) and for which \( \det(\nabla \tilde{g}_i(\chi^0, \psi^0)) \neq 0 \). Then, there exists a neighbourhood \( \Psi \) of \( \psi^0 \) and a unique function \( \phi_{i} : \chi = \phi_{i}(\psi) \) with \( \phi_{i} : \Psi \to \mathbb{R}^2 \) being continuously differentiable on \( \Psi \), such that \( \chi^0 = \phi_{i}(\psi^0) \) and \( \tilde{g}_i(\phi_{i}(\psi), \psi) = 0, \forall \psi \in \Psi \). In addition, the following result holds

\[
\nabla_{\psi} \phi_{i} = -(\nabla_{\chi} \tilde{g}_i)^{-1} \nabla_{\chi} \tilde{g}_i.
\]

Using Lemma B.1 for the implicit functions defined in (B5), it is now possible to prove, as desired, that for any \( i \in I_N \)

\[
\tilde{g}_0 \in \mathbb{R}^+, M_1 \in \mathbb{R}^+ : \| \epsilon(t_i^-) \| < \delta_0 \Rightarrow \left\| \Delta t_i \right\|_{\Delta \theta_i} \leq M_1 \| \epsilon(t_i^-) \|. \tag{25}
\]

First of all, in order to apply the implicit function theorem, all its hypothesis have to be verified. Taking \( \chi^0 = 0 \) and \( \psi^0 = 0 \), that is \( \Delta t_i = 0, \Delta \theta_i = 0 \) and \( e(t_i^-) = 0 \), one has \( \tilde{g}_i(\chi^0, \psi^0) = 0 \), for any \( i \in I_N \). In fact, such a choice yields in both cases the following results.

**Case a)** Setting \( \Delta t_i = 0, \Delta \theta_i = 0 \) and \( e(t_i^-) = 0 \) in (B3) one obtains

\[
(\chi^0, \psi^0) = 0 \text{ in (B4) one obtains}
\]

\[
\tilde{g}_1(0,0,0) = g_1(x_1^0, y_1^0, 0) = x_a(t_i^0)/a^2 + y_a(t_i^0)/b^2 - 1, \quad \tilde{g}_2(0,0,0) = g_2(x_1^0, y_1^0, 0) = \sin(\theta_{i}^0)x_a(t_i^0) - \cos(\theta_{i}^0)y_a(t_i^0),
\]

and since \( t_i^0 = i, i \in \mathbb{Z} \) are impact times for the desired trajectory, then \( \tilde{g}_1(0,0,0) = 0 \) and \( \tilde{g}_2(0,0,0) = 0 \) for any \( i \in I_N \), that is \( \tilde{g}_i(0,0,0) = 0 \). The Jacobian of \( \hat{g}_i \) with respect to \( \chi := [\Delta t_i, \Delta \theta_i]^T \) at the point \( (\chi^0, \psi^0) \) can be computed as

\[
\nabla_{\chi} \tilde{g}_i(\chi^0, \psi^0) = \begin{bmatrix} \frac{\partial \tilde{g}_1}{\partial \Delta t_i} & \frac{\partial \tilde{g}_1}{\partial \Delta \theta_i} \\ \frac{\partial \tilde{g}_{2,i}}{\partial \Delta t_i} & \frac{\partial \tilde{g}_{2,i}}{\partial \Delta \theta_i} \end{bmatrix}(\chi^0, \psi^0),
\]

where considering again that \( t_i^0 = i, i \in I_N \) are impact times for the desired trajectory, the following results hold

\[
\frac{\partial \tilde{g}_1}{\partial \Delta t_i}(\chi^0, \psi^0) = \frac{2}{a^2} x_a(t_i^0) y_a(t_i^0) + \frac{2}{b^2} y_a(t_i^0) x_a(t_i^0) \neq 0, \quad \frac{\partial \tilde{g}_1}{\partial \Delta \theta_i}(\chi^0, \psi^0) = 0,
\]

\[
\frac{\partial \tilde{g}_{2,i}}{\partial \Delta t_i}(\chi^0, \psi^0) = \cos(\theta_{i}^0)x_a(t_i^0) + \sin(\theta_{i}^0)y_a(t_i^0) \neq 0, \quad \frac{\partial \tilde{g}_{2,i}}{\partial \Delta \theta_i}(\chi^0, \psi^0) = 0,
\]

and this implies: \( \det(\nabla_{\chi} \tilde{g}_i(\chi^0, \psi^0)) \neq 0 \).
From Lemma B.1 it follows that, for any \( i \in \mathcal{I}_N \), the functions \( \tilde{g}(\Delta t_i, \Delta \theta_i, e(t_i^{m-})) \) implicitly define \( \mathbf{z} = \phi_i(\mathbf{z}) \) in a neighbourhood of \( \mathbf{z} = 0 \). In other terms, there exists a neighbourhood \( \Psi \) of \( e(t_i^{m-}) = 0 \) such that on that neighbourhood one can write

\[
\begin{bmatrix}
\Delta t_i \\
\Delta \theta_i
\end{bmatrix} = \phi_i(e(t_i^{m-})), \quad \text{for any } e(t_i^{m-}) \in \Psi,
\]

and the functions \( \phi_i(\cdot) \) are continuously differentiable.

Hence, from the well known Weierstrass theorem (see, e.g., Apostol (1974)) there exist \( N \) constants \( M_{1,i} \in \mathbb{R}^+ \) such that \( \|\nabla \phi_i(\mathbf{z})\| \leq M_{1,i} \) on the closure of \( \Psi \), and hence also on \( \Psi \), for any \( i \in \mathcal{I}_N \). Using this fact, a first order approximation for the functions \( \phi_i(\cdot) \) can be considered on a neighborhood of \( \mathbf{z} = 0 \) as

\[
\begin{bmatrix}
\Delta t_i \\
\Delta \theta_i
\end{bmatrix} = \phi_i(\mathbf{z}) + \nabla \phi_i(\mathbf{z}) \mathbf{z} + o(\|\mathbf{z}\|),
\]

where \( \mathbf{z} := e(t_i^{m-}) \). At this point, (25) is proved for any \( i \in \mathcal{I}_N \), defining \( M_i := \max_{i \in \mathcal{I}_N} \{M_{1,i}\} \) and considering a sufficiently small neighbourhood of \( e(t_i^{m-}) = 0 \) of radius \( \delta_0 \in \mathbb{R}^+ \).

### B.2 Details of the proof of fact (26)

For any \( i \in \mathcal{I}_N \), one has to prove that

\[
\exists \delta_1 \in \mathbb{R}^+, M_2 \in \mathbb{R}^+, M_3 \in \mathbb{R}^+: \|e(t_i^{m-})\| < \delta_1, \quad \left\| \begin{bmatrix}
\Delta t_i \\
\Delta \theta_i
\end{bmatrix} \right\| < \delta_1 \Rightarrow \|e(t_i^{M+})\| \leq M_2 \|e(t_i^{m-})\| + M_3 \left\| \begin{bmatrix}
\Delta t_i \\
\Delta \theta_i
\end{bmatrix} \right\|. \tag{26}
\]

The two possible cases are considered separately.

**Case a)** \( t_i^m = t_i^d, t_i^M = t_i^+ \), so that \( \Delta t_i < 0 \), and the error at time \( t_i^{M+} \), that is after the \( i \)-th couple of impacts, is given by \( e(t_i^{M+}) := z(t_i^{M+}) - z_d(t_i^{M+}) \), where

\[
\begin{align*}
\mathbf{z}(t_i^{M+}) &= e_{\tilde{A}t_i^{m-}}(t_i^{m-}) \mathbf{z}(t_i^{m-}) \\
&= e_{\tilde{A}t_i^{m-}}(t_i^{m-}) \mathbf{z}(t_i^{m-}) + \mathbf{A}_d(t_i^{M+}) \\
&= e_{\tilde{A}t_i^{m-}}(t_i^{m-}) \mathbf{z}(t_i^{m-}) + \mathbf{A}_d(t_i^{M+}) \\
\mathbf{z}_d(t_i^{M+}) &= C(\theta_i^d) \mathbf{z}_d(t_i^{m-}) + \mathbf{A}_d(t_i^{M+}) \\
&= C(\theta_i^d) \mathbf{z}_d(t_i^{m-}) + \mathbf{A}_d(t_i^{M+}).
\end{align*}
\]

Hence, it follows that

\[
\begin{align*}
e(t_i^{M+}) &= e_{\tilde{A}t_i^{m-}}(t_i^{m-}) C(\theta_i)(e(t_i^{m-}) + \mathbf{z}_d(t_i^{m-})) \\
&+ e_{\tilde{A}t_i^{m-}}(t_i^{m-}) \mathbf{A}_d(t_i^{M+}) - C(\theta_i^d) \mathbf{z}_d(t_i^{m-}) - \mathbf{A}_d(t_i^{M+}) \\
&= (e_{\tilde{A}t_i^{m-}}(t_i^{m-}) - 1) \mathbf{A}_d(t_i^{M+}) + (e_{\tilde{A}t_i^{m-}}(t_i^{m-}) - 1) C(\theta_i)(e(t_i^{m-}) + \mathbf{z}_d(t_i^{m-})) \\
&- C(\theta_i^d) e_{\tilde{A}t_i^{m-}}(t_i^{m-}) \mathbf{z}_d(t_i^{m-}) + e_{\tilde{A}t_i^{m-}}(t_i^{m-}) \mathbf{A}_d(t_i^{M+}) \\
&= (e_{\tilde{A}t_i^{m-}}(t_i^{m-}) - 1) \mathbf{A}_d(t_i^{M+}) + (e_{\tilde{A}t_i^{m-}}(t_i^{m-}) - 1) C(\theta_i)(e(t_i^{m-}) + \mathbf{z}_d(t_i^{m-})) \\
&- C(\theta_i^d) e_{\tilde{A}t_i^{m-}}(t_i^{m-}) \mathbf{z}_d(t_i^{m-}) + e_{\tilde{A}t_i^{m-}}(t_i^{m-}) \mathbf{A}_d(t_i^{M+}).
\end{align*}
\]

where \( \Delta \theta_i := \theta_i - \theta_i^d \);

**Case b)** \( t_i^m = t_i^d, t_i^M = t_i^+ \), so that \( \Delta t_i > 0 \), and the error at time \( t_i^{M+} \), that is after the \( i \)-th couple of impacts, is given by \( e(t_i^{M+}) := z(t_i^{M+}) - z_d(t_i^{M+}) \), where

\[
\begin{align*}
z(t_i^{M+}) &= C(\theta_i) z(t_i^{M+}) + \mathbf{A}_d(t_i^{m-}) \\
&= e_{\tilde{A}t_i^{m-}}(t_i^{m-}) \mathbf{z}(t_i^{m-}) + \mathbf{A}_d(t_i^{m-}) \\
&= e_{\tilde{A}t_i^{m-}}(t_i^{m-}) \mathbf{z}(t_i^{m-}) + \mathbf{A}_d(t_i^{m-}) \\
\mathbf{z}_d(t_i^{M+}) &= C(\theta_i^d) \mathbf{z}_d(t_i^{m-}) + \mathbf{A}_d(t_i^{m-}) \\
&= C(\theta_i^d) \mathbf{z}_d(t_i^{m-}) + \mathbf{A}_d(t_i^{m-}).
\end{align*}
\]

Hence, it follows that

\[
\begin{align*}
e(t_i^{M+}) &= C(\theta_i) z(t_i^{M+}) + \mathbf{A}_d(t_i^{m-}) - e_{\tilde{A}t_i^{m-}}(t_i^{m-}) C(\theta_i^d) \mathbf{z}_d(t_i^{m-}) \\
&- e_{\tilde{A}t_i^{m-}}(t_i^{m-}) \mathbf{A}_d(t_i^{M+}) \\
&= (1 - e_{\tilde{A}t_i^{m-}}(t_i^{m-})) \mathbf{A}_d(t_i^{m-}) + (C(\theta_i^d) e_{\tilde{A}t_i^{m-}}(t_i^{m-}) \\
&- e_{\tilde{A}t_i^{m-}}(t_i^{m-}) C(\theta_i^d) \mathbf{z}_d(t_i^{m-}) + C(\theta_i^d) \mathbf{A}_d(t_i^{m-})) \\
&= (1 - e_{\tilde{A}t_i^{m-}}(t_i^{m-})) \mathbf{A}_d(t_i^{m-}) + (C(\theta_i^d) + \Delta \theta_i) e_{\tilde{A}t_i^{m-}}(t_i^{m-}) \\
&- e_{\tilde{A}t_i^{m-}}(t_i^{m-}) C(\theta_i^d) \mathbf{z}_d(t_i^{m-}) + C(\theta_i^d + \Delta \theta_i) e_{\tilde{A}t_i^{m-}}(t_i^{m-}),
\end{align*}
\]

where \( \Delta \theta_i := \theta_i - \theta_i^d \);

It is clear that, for any \( i \in \mathcal{I}_N \), in both cases a) and b), that is for \( \Delta t_i < 0 \) and \( \Delta t_i > 0 \), respectively, the functions \( e(t_i^{M+}) \) are equal to \( 0 \) when \( \Delta t_i = 0, \Delta \theta_i = 0 \) and \( e(t_i^{m-}) = 0 \) and they are analytic functions with respect to their variables. In view of this fact, for each case separately the following results hold

**Case a)**

\[
\exists \delta_1 \in \mathbb{R}^+, M_{2,a,i} \in \mathbb{R}^+, M_{3,a,i} \in \mathbb{R}^+: \|e(t_i^{m-})\| < \delta_1, \quad -\delta_1 < \Delta t_i < 0, |\Delta \theta_i| < \delta_1 \\
\Rightarrow \|e(t_i^{M+})\| \leq M_{2,a,i} \|e(t_i^{m-})\| + M_{3,a,i} \left\| \begin{bmatrix}
\Delta t_i \\
\Delta \theta_i
\end{bmatrix} \right\|.
\]
Case b)
\[ \exists \delta_1 \in \mathbb{R}^+, M_{2,b,j} \in \mathbb{R}^+, M_{1,b,j} \in \mathbb{R}^+ : \| e(t_i) \| < \delta_1, 0 < \Delta t_i < \delta_1, | \Delta \theta_i | < \delta_1 \]
\[ \Rightarrow \| e(t_i) \| \leq M_{2,b,j} \| e(t_i) \| + M_{3,b,j} | \Delta \theta_i | \]

Moreover, it is easy to see that, for any \( i \in \mathcal{I}_N \), the functions \( e(t_i) \) are continuous on the hyperplane characterized by \( \Delta t_i = 0 \) (it is also easy to show that \( e(t_i) \) are differentiable at \( \Delta t_i = 0, \Delta \theta_i = 0, e(t_i) = 0 \). In particular, in both cases a) and b), one has
\[ e(t_i) \mid_{\Delta t_i=0} = (C(\theta_i^d + \Delta \theta_i) - C(\theta_i^d)z, t_i - \theta_i^d) + C(\theta_i^d + \Delta \theta_i)e(t_i)^{d+1} \]

Therefore, (26) are proved, for any \( i \in \mathcal{I}_N \), choosing
\[ M_2 := \max \{ M_{2,b,j} \} \text{ and } M_3 := \max \{ M_{3,b,j} \} \]

B.3 Details of the proof of fact (27)

During the unconstrained motion, from (17) and (19) it follows that \( \dot{z}(t) = Az(t) + Bu(t) \) and \( \dot{z}_d = A_{x\theta}(t) \), where \( A := \text{blockdiag}(A, A) \in \mathbb{R}^{2(q+1) \times 2(q+1)}, B := \text{blockdiag}(B, B) \in \mathbb{R}^{2(q+1) \times 2} \) and \( u(t) := [u_{x_0}, u_{x_1}]^T \in \mathbb{R}^2 \).

Using the control law given in (23) and defining \( K := \text{blockdiag}(K, K) \in \mathbb{R}^{2(q+1) \times 2(q+1)} \) the error dynamics of the closed-loop system is given by

\[
\dot{e}(t) = (\hat{A} + BK)e(t) = \hat{A}e(t),
\]
\[ \forall t \in (t_i^M, t_i^{m+1}), i \in \mathcal{I}, i \geq [t_0], \quad (B6) \]

where all eigenvalues of \( \hat{A} \) have real part less than or equal to \( -\eta \). Hence, in any interval of time without impacts one has
\[ \exists L \in \mathbb{R}^+ : \| e(t) \| \leq L \eta^{(t-t_0)(t)} \| e(t_i) \|, \quad \forall t \in (t_i^M, t_i^{m+1}), i \in \mathcal{I}, i \geq [t_0], \]

In order to prove
\[ \forall \eta \in \mathbb{R}^+ : \| e(t) \| \leq L(\eta) \exp(-\eta(t-t_0)(t)) \| e(t_i) \|, \forall t \in (t_i^M, t_i^{m+1}), i \in \mathcal{I}, i \geq [t_0], \text{ with } \lim_{\eta \rightarrow +\infty} L(\eta) \exp(-\eta(t-t_0)(t)) = 0, \forall T > 0, \]

the error dynamics given in (B6) can be rewritten in terms of \( e_{x_0} := x_{x_0} - x_{x_0} \) and \( e_{y_0} := y_{y_0} - y_{y_0} \) as
\[
\begin{cases}
\dot{e}_{x_0} = (A + BK_x)e_{x_0} = \tilde{A}_x e_{x_0}, \\
\dot{e}_{y_0} = (A + BK_y)e_{y_0} = \tilde{A}_y e_{y_0},
\end{cases}
\]

where the matrix \( \tilde{A}_x \) is in companion form (for the sake of brevity, the case \( e_y \) is not considered being analogous to \( e_x \))

\[
\tilde{A}_x = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0 \\
K_1 & \cdots & K_{q+1}
\end{bmatrix}
\]

and \( K_x := [K_1^x, \ldots, K_{q+1}^x] \), \( K_i \in \mathbb{R}, i \in [1, \ldots, q+1] \).

If the eigenvalues of \( \tilde{A}_x \) denoted as \( \lambda_1, \lambda_2, \ldots, \lambda_{q+1} \), are all real and distinct, then it is well known (Chen 1984) that the eigenvectors relative to such eigenvalues are given by

\[
\begin{bmatrix}
1 \\
\lambda_1 \\
\lambda_1^2 \\
\vdots \\
\lambda_1^q
\end{bmatrix}
\]

(TThe case of complex eigenvalues is not considered here for the sake of brevity. It can be carried out with similar reasons, by considering \( \lambda_i = -\mu_i \eta + j\omega_i \) in place of (B7), where \( \mu_i, \eta, \omega \in \mathbb{R} \) and \( \mu_i \geq 1, \eta > 0 \).

Without loss of generality, the \( i \)th eigenvalue can be written as

\[
\lambda_i = -\mu_i \eta_i, \quad (B7)
\]

where \( \eta, \mu, \eta > 0, \mu_1 \geq 1 \) with \( \mu_i \neq \mu_j \) for any \( i \neq j \), so that \( \lambda_i \leq -\eta \) for any \( i \in [1, q+1] \). The matrix of eigenvectors is then given by

\[
Q_x := \begin{bmatrix}
1 & 1 & \cdots & 1 \\
(-\mu_1 \eta)^2 & (-\mu_2 \eta)^2 & \cdots & (-\mu_{q+1} \eta)^2 \\
(-\mu_1 \eta)^q & (-\mu_2 \eta)^q & \cdots & (-\mu_{q+1} \eta)^q
\end{bmatrix},
\]

and since the eigenvalues are all distinct, then \( \det(Q_x) \neq 0 \). Considering the infinity norm defined as \( \| Q_x \| := \max_{i\in[1, q+1]} \sum_{j=1}^{q+1} |Q_x(i,j)| \), where \( Q_x(i,j) \) denotes the \( j \)th element of \( Q_x \), and by the equivalence norms (see, e.g., Griffel (2002)), it is easy to see that
\[ \exists C_1 \in \mathbb{R}^+ : \| Q_x \| \leq C_1 p_1(\eta), \]
where \( p_1(\eta) := \sum_{\nu(i,j)} |Q_x(i,j)| \) is a polynomial in \( \eta \) of degree \( q \). The inverse matrix of \( Q_x \) can be computed as

\[
Q_x^{-1} = \frac{1}{\det(Q_x)} \text{Adj}(Q_x).
\]

In view of the particular structure of the matrix \( Q_x \), it is not difficult to show that \( \det(Q_x) = L_1(\eta q^{(q+1)/2}) \), \( L_1 \in \mathbb{R} \) and the generic element of \( \text{Adj}(Q_x) \) is given by \( \text{Adj}(Q_x)(i,j) = L_2^{(i,j)} \eta^{((q+1)/2) - (j-1)} \), \( L_2^{(i,j)} \in \mathbb{R} \) where \( i,j \in \{1, \ldots, q+1\} \). Therefore, the \( j \)th element of \( Q_x^{-1} \) can be written as \( Q_x^{-1}(i,j) = L_3^{(i,j)} \eta^{-j} \), \( L_3^{(i,j)} \in \mathbb{R} \) with \( i,j \in \{1, \ldots, q+1\} \), so that

\[
\|Q_x^{-1}\| \leq C_1 p_2(\eta^{-1}),
\]

where \( p_2(\eta^{-1}) := \sum_{\nu(i,j)} |Q_x^{-1}(i,j)| \) is a polynomial in \( \eta^{-1} \) of degree \( q \). Finally, the following upper bound for the condition number of \( Q_x \) holds

\[
\|Q_x\|\|Q_x^{-1}\| \leq C_1^2 p_1(\eta) p_2(\eta^{-1}) =: L(\eta),
\]

where \( L(\eta) \) is a polynomial in \( \eta \) of degree \( q \). This result implies that

\[
\|e^{(A+BK)_T}\|
\leq \|Q_x\|\|\text{diag}(e^{-\mu_1 T}, e^{-\mu_2 T}, \ldots, e^{-\mu_q T})\|\|Q_x^{-1}\|
\leq e^{-\eta T}L(\eta) \to 0 \quad \text{as} \quad \eta \to +\infty, \quad \text{for any fixed} \quad T > 0,
\]

and this complete the proof of (27) when all the eigenvalues of \( A_x \) are real and distinct. In a similar way, the bound (B8) can be proved in the case of repeated real eigenvalues, taking into account that if \( \lambda_i \) is an eigenvalue of the matrix \( \bar{A}_x \), with multiplicity \( m_i \), then there exist \( m_i \) generalized eigenvectors of \( \bar{A}_x \) associated with \( \lambda_i \); \( \bar{e}_1^{(i)}, \ldots, \bar{e}_m^{(i)} \), such that the \( k \)th element of the \( j \)th of such eigenvectors is given by

\[
e_j^{(i)}(k) = \left( \frac{k-1}{j-1} \right) \lambda_i^{j-k},
\]

where \( k, \ldots, q+1, j = 1, \ldots, m_i \) and

\[
\binom{k}{j} := \begin{cases} 0, & \text{if } k < j; \\ 1, & \text{if } k = j; \\ k(k-1) \cdots (k+1-j) \cdot 1,2,3,\ldots,j, & \text{otherwise}. \end{cases}
\]


