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Strong stabilization with infinite multivariable gain margin through linear periodic control

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The problem of the strong stabilization with infinite gain margin, with the additional requirement of a prescribed rate of convergence of the free responses, is addressed for linear time-invariant discrete-time multivariable plants in the case when unknown different scalar gains act either on the inputs or on the outputs. Necessary and sufficient conditions for the solvability of the problem by means of a stable linear periodic discrete-time output feedback dynamic controller are derived. Algorithmic procedures are given for designing the proposed periodic controllers.

1. Introduction

The gain margin problem, in the SISO case, is the problem of finding a compensator that is able to stabilize the series connection of an unknown gain $k$ and a linear time invariant plant $S$, for every $k \in [a, b]$, $a, b \in \mathbb{R}^+$; this problem was studied and solved by means of interpolation theory in Tannenbaum (1980, 1982) and Khargonekar and Tannenbaum (1985) for the case when the controller is linear and time-invariant. If the unknown scalar gain is allowed to vary over an infinite interval $[a, +\infty)$, the problem is called the infinite gain margin (IGM) problem. Periodic controllers to obtain an arbitrarily large gain margin were described in Khargonekar et al. (1985) for discrete-time plants, in Lee et al. (1987) for continuous-time plants.

For MIMO systems, the gain margin problem can be extended in two different ways: by considering a scalar unknown constant gain acting on every input (or output) of the system, or by considering different independent unknown constant gains acting on each scalar input (or each scalar output) of the system. Note that, in the second case, in general a controller designed to solve the problem when the unknown gains act on the input is not able to solve the problem when the unknown gains act on the output, and vice versa.

An arbitrarily large gain margin for the first kind of extension by linear periodic controllers was obtained in Francis and Georgiou (1988) and Yan et al. (1994) both for discrete-time systems and for continuous-time systems by digital control schemes (in Francis and Georgiou 1988) by means of a conventional periodic digital controller, in Yan et al. (1994) by the use of generalized sampled-data hold functions (GSHF).

For the second kind of extension, in the case when the unknown gains act on the inputs of the plant, the authors are aware of two kinds of results valid for continuous-time systems: the well-known LQR, state feedback results (see, e.g. Wonham 1979, Dorato et al. 1995) and some sufficient conditions (which become necessary if decoupling is also required) in Maeda and Vidyasagar (1985) for dynamic output feedback.

The strong stabilization problem, that is the problem of finding an asymptotically stable controller to stabilize an unstable plant, has been studied by several authors (see, e.g. Youla et al. 1974, Vidyasagar 1985 for continuous-time plants). By using linear time-invariant controllers, the problem can be solved if and only if a parity interlacing property (PIP) between the real blocking zeros and the real poles of the system to be controlled is satisfied. However, since blocking zeros can be removed (in a suitable sense) by periodic compensation, several solutions exist for stabilizable and detectable plants not satisfying the PIP if linear periodic controllers are used: specifically, digital periodic controllers (Khargonekar et al. 1985), multirate controllers (Mita et al. 1987, Hagiwara and Araki 1988, In and Zhang 1994, Er and Anderson 1995), GSHF controllers (Chammas and Leondes 1978, Kabamba 1987) have been proposed in the literature to solve the strong stabilization problem.

In this paper, a strong stabilization problem, requiring a prescribed rate of convergence of the free responses and an infinite gain margin (of the second type), is studied for discrete-time linear time invariant (LTI) MIMO plants. The two cases, where the unknown scalar gains act on the inputs or on the outputs of the systems, are considered. Necessary and sufficient conditions and design procedures will be given for the solution of the problem in the two cases, via an asymptotically stable (actually, with a dead-beat impulse response) linear periodic controller. It is worth mentioning that the necessary and sufficient conditions derived in this paper to solve the IGM problem of the second kind are actually weaker than the sufficient conditions...
available in the literature to solve the simpler IGM problem of the first kind (see the subsequent Remark 5).

**Notation:** In general, for a linear discrete-time system $\tilde{S}$, its input, state and output will be denoted by $u^\mathbf{1}(\cdot)$, $x^\mathbf{2}(\cdot)$ and $y^\mathbf{3}(\cdot)$, respectively, unless otherwise specified, and the matrices appearing in its state-space description will be denoted (unless otherwise specified) by $A^\mathbf{1}(\cdot)$, $B^\mathbf{2}(\cdot)$, $C^\mathbf{3}(\cdot)$ and $D^\mathbf{3}(\cdot)$, so that, if $k \in \mathbb{Z}$ is time, $\tilde{S}$ will be characterized by the equations

\[
\begin{align*}
\dot{x}(k+1) &= A^\mathbf{1}(k)x(k) + B^\mathbf{2}(k)u(k) \\
y(k) &= C^\mathbf{3}(k)x(k) + D^\mathbf{3}(k)u(k)
\end{align*}
\]

and the notation $\tilde{S} = (A^\mathbf{1}(\cdot), B^\mathbf{2}(\cdot), C^\mathbf{3}(\cdot), D^\mathbf{3}(\cdot))$ will be used. However, if a system $S$ similar to $\tilde{S}$, either LTI or linear periodically time-varying (LPTV), is non-dynamic, and therefore characterized by its only matrix $D^\mathbf{3}(\cdot)$, with an abuse of notation (in order to simplify it) such a matrix will be simply denoted by the same symbol $Q$ representing the system. Throughout the paper, $\mathbb{Z}^+$ and $\mathbb{R}^+$ will denote, respectively, the set of non-negative integers and the set of non-negative reals, whereas $\mathbb{Z}^*$ and $\mathbb{R}^*$ will denote, respectively, the set of positive integers and the set of positive reals.

The identity matrix of dimension $v$ will be denoted by $I_v$, or, when confusion cannot arise, simply by $I$. Zero vectors, of dimension $s$, and matrices, of dimension $s \times r$, will be denoted by $0_s$ and $0_{s,r}$, respectively, whenever specifying such dimensions will be useful. Further, given a vector $v \in \mathbb{R}^s$ and an ordered set of positive integers $s = [s_1, \ldots, s_e]$ of cardinality $e \leq a$, and such that $1 \leq s_1 < s_e \leq a$, two new vectors $(v)_{(i)} \in \mathbb{R}^r$ and $(v)_{(o)} \in \mathbb{R}^r$ are defined, where the $i$th component of $(v)_{(o)}$ is equal to the $s_i$th component of $v$ for each $i = 1, \ldots, e$, whereas $(v)_{(o)}$ is equal to $v$ with the exception of the components in positions $s_1, \ldots, s_e$ which are equal to zero. For singletons like $[s_1]$, the shorthand $(v)_{(s_1)}$ will be used instead of $(v)_{(1)}$. Given a matrix $M \in \mathbb{R}^{s \times b}$ and two ordered sets of positive integers $s = [s_1, \ldots, s_p]$ and $t = [t_1, \ldots, t_f]$ of cardinality $f \leq b$, and such that $1 \leq s_1 < s_p \leq a$, $1 \leq t_1 < t_f \leq b$, two new matrices $(M)_{(i,t)} \in \mathbb{R}^{s_p \times t_f}$ and $(M)_{(t,i)} \in \mathbb{R}^{s_p \times b}$ are defined, where the $(i,t)$ element of $(M)_{(i,t)}$ is equal to the element $(s_i, t_j)$ of $M$ for each $i = 1, \ldots, e$, $j = 1, \ldots, f$, whereas all the elements of $(M)_{(t,i)}$ are equal to the corresponding elements of $M$ with the exception of the elements in positions $(s_i, t_j)$ for each $i = 1, \ldots, e$, $j = 1, \ldots, f$ which are equal to zero. For singletons like $[s_1]$, the shorthand $(M)_{(s_1, i)}$ will be used instead of $(M)_{(1, i)}$. A similar notation will be used if $t$ or, possibly, both $s$ and $t$ are singletons.

For any given square matrix $M \in \mathbb{R}^{n \times n}$, the spectral radius of $M$ (i.e. the maximum of the moduli of the eigenvalues of $M$) will be denoted by $r(M)$. Note that the matrix norm $\|M\|_\infty$, computed as $\|M\|_\infty = \max_{i \in 1, \ldots, n} \sum_{j=1}^{n} |(M)|_{i,j}$, satisfies the relation $r(M) \leq \|M\|_\infty$ (Desoer and Vidyasagar 1975).

For any matrix $N \in \mathbb{R}^{n \times n}$ of rank $\min(\mu, v)$, $N^\perp$ will denote its generalized inverse (or pseudoinverse), given by $N^\perp = (N^*N)^{-1}N^*$ if $\nu \leq \mu$ and by $N^\perp = N^*(NN^*)^{-1}$, if $\mu \leq \nu$.

In the following, for $j, l \in \mathbb{Z}$, if $j < l$, then by definition $\sum_{i=j}^{l} = 0$, even when the argument of the sum cannot be computed for $i < j$ or for $i > l$.

## 2. Preliminaries and problem statement

The problems to be solved here, i.e. the subsequent Problems 1 and 2, will be stated by referring to an LTI system $S$, which will play the role of the nominal model of the plant to be controlled. The state space description of $S = (A, B, C, D)$ is characterized by the equations

\[
\begin{align*}
\dot{x}(k+1) &= Ax(k) + Bu(k) \\
y(k) &= Cx(k) + Du(k)
\end{align*}
\]

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^q$, $y(k) \in \mathbb{R}^q$, and $A, B, C$ and $D$ are constant.

In addition, since a periodic compensator will be used, consider an LPTV discrete-time system $\Theta = (A(k), B(k), C(k), D(k))$, described by

\[
\begin{align*}
\dot{x}(k+1) &= \tilde{A}(k)x(k) + \tilde{B}(k)\tilde{u}(k) \\
\tilde{y}(k) &= \tilde{C}(k)\tilde{x}(k) + \tilde{D}(k)\tilde{u}(k)
\end{align*}
\]

where $\tilde{x}(k) \in \mathbb{R}^{n \times \tilde{k}}$, $\tilde{u}(k) \in \mathbb{R}^{n \times \tilde{k}}$, $\tilde{y}(k) \in \mathbb{R}^{n \times \tilde{k}}$, and $A(k), B(k), C(k), D(k)$, are periodic matrices of period $\omega$, with entries with $\omega$-periodic entries with entries $\in \mathbb{R}$. Later on, the definitions and notations given for $\tilde{\Theta}$ will be referred to any LPTV system, e.g. the compensator to be synthesized. Specifically, from now on, denote by $\tilde{\Theta}(i, j)$ the state transition matrix of $\tilde{\Theta}$ from time $j$ to time $i$ (where $\tilde{\Theta}(i, j) = A(i-1)A(i-2)\cdots A(j)$ if $i > j$, and $\tilde{\Theta}(j, i) = I_{\tilde{k}}$). It is well known that, for any initial time $k_0 \in \mathbb{Z}$, and for all initial states $\tilde{x}(k_0) \in \mathbb{R}^{n \times \tilde{k}}$ and all input functions $\tilde{u}(\cdot)$, the output response $\tilde{y}(\cdot)$ of the system $\tilde{\Theta}$ for $k \geq k_0$ can be obtained from the output response of a lifted representation of $\tilde{\Theta}$, and specifically from that introduced in Meyer and Burris (1975), that is an LTI system $\tilde{\Theta}_{k_0}$ whose input and output sequences are the lifted representations of the input and output sequences of $\tilde{\Theta}$. The state space description of system $\tilde{\Theta}_{k_0}$ is expressed by

\[
\begin{align*}
\dot{\tilde{x}}_{k_0}(h+1) &= \tilde{A}_{k_0}\tilde{x}_{k_0}(h) + \tilde{B}_{k_0}\tilde{u}_{k_0}(h) \\
\tilde{y}_{k_0}(h) &= \tilde{C}_{k_0}\tilde{x}_{k_0}(h) + \tilde{D}_{k_0}\tilde{u}_{k_0}(h)
\end{align*}
\]
where \( h \in \mathbb{Z}^+ \), \( \bar{x}_k(h) \in \mathbb{R}^d \), \( \tilde{u}_k(h) \in \mathbb{R}^{q_d} \), \( \bar{y}_k(h) \in \mathbb{R}^{q_y} \),
\[ \bar{A}_k = \bar{\Phi}(k_0 + \omega, k_0), \quad \bar{B}_k = \bar{\Phi}(k_0 + \omega, k_0 + 1) \bar{B}(k_0) \ldots \bar{\Phi}(k_0 + \omega, k_0 + \omega) \bar{B}(k_0 + \omega), \quad \bar{C}_k = \{ (\bar{C}(k_0) \Phi(k_0), k_0) \ldots (\bar{C}(k_0 + \omega - 1) \Phi(k_0 + \omega - 1, k_0) \}_1 \}, \] and \( \bar{D}_k \) is a \((\omega \times \omega)\) block matrix whose \((i, j)\) block is zero if \( i < j \), is equal to \( \bar{D}(k_0 + j - 1, k_0 + j - 1) \) if \( i = j \) and is equal to \( \bar{C}(k_0 + i - 1) \Phi(k_0 + i, k_0 + j) \bar{B}(k_0 + j - 1) \) otherwise. In fact, if \( \bar{x}_k(0) = \bar{x}(k_0) \) and \( \tilde{u}_k(h) = \{ [u(k_0 + h) \ldots u(k_0 + h + 1) h]\}^T \) for all \( h \in \mathbb{Z}^+ \), then \( \bar{x}_k(h) = \bar{x}(k_0 + h) \) and \( \bar{y}_k(h) = \{ [y(k_0 + h) \ldots y(k_0 + h + 1) h]\}^T \) for all \( h \in \mathbb{Z}^+ \). From now on, \( \bar{\Theta}_k \) will be called the associated system of \( \bar{\Theta} \) at \((\text{the initial})\) time \( k_0 \).

Now consider the special case when system \( \bar{\Theta} \) is time invariant, i.e. matrices \( \bar{A}(\cdot) \), \( \bar{B}(\cdot) \), \( \bar{C}(\cdot) \) and \( \bar{D}(\cdot) \) are constant, as in the case of the previously mentioned system \( S = (A, B, C, D) \). One can always look at system \( S \) as an LPTV system of arbitrary period \( \omega \geq 1 \); moreover, for each choice of the period \( \omega \), the associated systems of \( S \) at all initial times \( k_0 \) have the same state space description as \( S_0 = (A_0, B_0, C_0, D_0) \) where \( A_0 = A^\omega \), \( B_0 = [A^{\omega-1} B \ldots B] \), \( C_0 = [C' \ldots (CA^{\omega-1})]' \), and \( D_0 \) a \((\omega \times \omega)\)-block matrix whose \((i, j)\) block is 0 if \( i < j \), is equal to \( D \) if \( i = j \) and is equal to \( CA^{i-j} B \) if \( i > j \).

In the following, an LPTV system (with period \( \omega > 1 \)) is denoted by a tilded capital letter, while an LTI system is denoted by a capital letter with no tilde. However, the LTI associated system of an LPTV system (like \( \bar{\Theta} \)) at time \( k_0 \) will be denoted by the tilded capital letter denoting the LPTV system with the time \( k_0 \) as subscript (like \( \bar{\Theta}_{k_0} \)).

In addition, recall that the characteristic multipliers of \( \bar{A}(\cdot) \), i.e. the eigenvalues of \( \bar{A}_k \), are independent of \( k_0 \), together with their algebraic multiplicities in the characteristic polynomial of \( \bar{A}_k \) (Evans 1972); therefore they characterize the asymptotic stability of \( \bar{\Theta} \), and also the rate of convergence of the state free motions of \( \bar{\Theta} \), according to the following definition.

**Definition 1:** For a given real positive constant \( \rho \leq 1 \), the LPTV system \( \bar{\Theta} \) of period \( \omega \) is said to be \( \rho \)-stable if there exist a positive \( \bar{\rho} < \rho \) and a positive \( \omega \)-periodic \( \tau(k_0) \) such that for all initial times \( k_0 \), and for all initial states \( \bar{x}(k_0) \in \mathbb{R}^d \), the state free motions of \( \bar{\Theta} \) satisfy the relation
\[
\| \bar{x}(k) \| < \tau(k_0) \rho \| \bar{x}(k_0) \| \quad \forall k \geq k_0.
\]

In fact, by applying this definition—whose form allows us to take into account the mere asymptotic stability as a special case for \( \rho = 1 \)—to the LTI system \( \Theta_{k_0} \) described by (4), whose period is 1 and whose time variable is \( h \), it is easy to derive the following proposition.

**Proposition 1:** The \( \omega \)-periodic system \( \bar{\Theta} \) is \( \rho \)-stable if and only if all the characteristic multipliers of \( \bar{A}(\cdot) \) are smaller than \( \rho^\omega \), in modulus, or, equivalently, if and only if, for an arbitrary \( k_0 \), \( \Theta_{k_0} \) is \( \rho^\omega \)-stable.

In the following, the characteristic multipliers of \( \bar{A}(\cdot) \) will also be called the characteristic multipliers of system \( \bar{\Theta} \). Since the purpose of this paper is to make use of an LPTV dynamic compensator in order to control an LTI system, all the previous definitions, notation and discussion concerning the LPTV system \( \bar{\Theta} \) can be applied to the LPTV connection of an LPTV (sub)compensator and an LTI subsystem. In particular, when referring to the LTI system \( S \) by itself (whose period \( \omega = 1 \)), its characteristic multipliers are the eigenvalues of \( A \) (whose modulus is to be smaller than \( \rho \) for the \( \rho \)-stability of \( S \)).

The LTI system \( S \) will be said to be \( \rho \)-stabilizable if there exists a linear map \( K \) such that all the eigenvalues of \( A + BK \) are smaller than \( \rho \), in modulus; it will be said to be \( \rho \)-detectable if a dual property holds. It is recalled that \( \rho \)-stabilizability and \( \rho \)-detectability of \( S \) can be checked, respectively, by the conditions (Grasselli and Longhi 1991)
\[
\text{rank} \left[ z I_n - A \ B \right] = n, \quad \forall z \in \mathbb{C}, \quad |z| \geq \rho \quad (5)
\]
and
\[
\text{rank} \left[ z I_n - A \ C \right] = n, \quad \forall z \in \mathbb{C}, \quad |z| \geq \rho. \quad (6)
\]

It is also recalled that controllability and reconstructability of \( S \) can be tested by checking the rank of the matrices in (5) and (6), respectively, for all non-zero \( z \in \mathbb{C} \) (Grasselli 1980). The properties of \( \rho \)-stabilizability and \( \rho \)-detectability (as well as controllability and reconstructability) of an LPTV system can be similarly defined and checked by completely analogous conditions (Grasselli and Longhi 1991).

This paper studies the strong \( \rho \)-stabilization with infinite gain margin for input perturbations (respectively, output perturbations), for a family \( S_{\infty} \) (respectively, \( S_{\text{out}} \)) of LTI systems to be controlled, as defined in Definition 3 (respectively, Definition 4), on the basis of the nominal model \( S \) of the plant.

**Definition 2:** For any \( m \in \mathbb{Z}^+ \), the family \( \lambda_m \) of admissible multiplicative perturbations for \( m \)-dimensional signals is defined as \( \lambda_m = \{ \Xi \in \mathbb{R}^{m \times m} \mid \Xi = \text{diag}(\xi_1, \ldots, \xi_m), \xi_i \in [1, +\infty), i = 1, \ldots, m \} \).

**Definition 3:** For the nominal system \( S = (A, B, C, D) \), the family \( S_{\text{in}} \) of input perturbed LTI plants \( S_{\text{in}} \) is defined as the set of the LTI systems obtained as the cascade connection \( S_{\text{in}} \) of an admissible multiplicative perturbation \( \Xi \in \lambda_m \) and \( S \), that is \( S_{\text{in}} = \{ S_{\text{in}} = (A_{\text{in}}, B_{\text{in}}, C_{\text{in}}, D_{\text{in}}) \mid A_{\text{in}} = A, B_{\text{in}} = B \Xi, C_{\text{in}} = C, D_{\text{in}} = D \Xi, \Xi \in \lambda_m \} \).

**Definition 4:** For the nominal system \( S = (A, B, C, D) \), the family \( S_{\text{out}} \) of output perturbed LTI plants \( S_{\text{out}} \) is defined as the set of the LTI systems obtained as the cascade connection \( S_{\text{out}} \) of \( S \) and an admissible
multiplicative perturbation \( \Xi \in \mathcal{X}_p \), that is, \( S_{\text{out}} = \{ S_{\text{out}}^\Xi = (A_{\text{out}}^\Xi, B_{\text{out}}^\Xi, C_{\text{out}}^\Xi, D_{\text{out}}^\Xi) : A_{\text{out}}^\Xi = A, B_{\text{out}}^\Xi = B, C_{\text{out}}^\Xi = \Xi C, D_{\text{out}}^\Xi = \Xi D, \Xi \in \mathcal{X}_p \} \).

**Remark 1:** Throughout the paper the output the paper the output \( y_{\text{out}}^\Xi = y_S^\Xi \) or \( y_{\text{out}}^\Xi = y^3 \) of the actual plant to be controlled \( S_{\Xi}^\text{out} \) or \( S_{\text{out}}^\Xi \), respectively, will be assumed to be its controlled output and will be denoted simply by \( y \).

Analogously, the input \( u_{\text{out}}^\Xi = u^\Xi \) or \( u_{\text{out}}^\Xi = u^3 \) of the actual plant to be controlled \( S_{\Xi}^\text{out} \) or \( S_{\text{out}}^\Xi \), respectively, will be denoted simply by \( u \).

**Remark 2:** Note that the choice of the interval \([1, +\infty)\) for the admissible scalar gains \( \xi \) in Definition 2 does not imply any loss of generality. As a matter of fact, if more general intervals of the kind \([\xi_{i,0}, +\infty)\), with \( \xi_{i,0} \neq 1 \) (and \( \xi_{i,0} > 0 \)), were to be considered, it would be sufficient to modify the nominal plant \( S \) by multiplying either the \( i^\text{th} \) column of \( B \) and \( D \) (in the case of input perturbations) or the \( i^\text{th} \) row of \( C \) and \( D \) (in the case of output perturbations) by \( \xi_{i,0} \).

In order to avoid trivial cases, the following assumption on the nominal system \( S \) is made, for the case of input perturbations.

**Assumption 1:** The nominal system \( S = (A, B, C, D) \) is such that for any matrix \( B^0 \) composed by \((p-1)\) different columns of \( B \), the pair \((A, B^0)\) is not \( \rho \)-stabilizable.

The role of Assumption 1 for the case of output perturbations is played by the following assumption.

**Assumption 2:** The nominal system \( S = (A, B, C, D) \) is such that for any matrix \( C^0 \) composed by \((q-1)\) different rows of \( C \), the pair \((A, C^0)\) is not \( \rho \)-detectable.

The problem considered here of the strong \( \rho \)-stabilization and infinite gain margin can be formally stated for the case of input perturbations.

**Problem 1** (strong infinite gain margin \( \rho \)-stabilization for input perturbations): For the given nominal system \( S \) satisfying Assumption 1, and for a prescribed positive \( \rho \leq 1 \), find (if any) a period \( \omega \geq 1 \) and an asymptotically stable LPTV dynamic output feedback compensator \( \hat{K} \) of period \( \omega \) such that the closed-loop LPTV control system \( \hat{\Sigma} \) obtained by connecting \( \hat{K} \) and \( S_{\text{in}}^\Xi \) is well-posed and \( \rho \)-stable for all \( S_{\Xi}^\text{out} \in S_{\text{out}} \), i.e. for all \( \Xi \in \mathcal{X}_p \).

**Remark 3:** If Assumption 1 is violated, a smaller number \( \hat{p} < p \) of inputs might be used in order to \( \rho \)-stabilize \( S_{\text{in}}^\Xi \), since the infinite gain margin is always guaranteed for the unused inputs (which can be all inputs if \( S \) is \( \rho \)-stable). In this case, however, the choice of which inputs to use is to be made with care, in order to satisfy Assumption 1 and condition (ii) of the subsequent Theorem 1.

The dual problem, for the case of output perturbations, is the following.

**Problem 2** (strong infinite gain margin \( \rho \)-stabilization for output perturbations): For the given nominal system \( S \) satisfying Assumption 2, and for a prescribed positive \( \rho \leq 1 \), find (if any) a period \( \omega \geq 1 \) and an asymptotically stable LPTV dynamic output feedback compensator \( \hat{K} \) of period \( \omega \) such that the closed-loop LPTV control system \( \hat{\Sigma} \) obtained by connecting \( \hat{K} \) and \( S_{\text{in}}^\Xi \) is well-posed and \( \rho \)-stable for all \( S_{\Xi}^\text{out} \in S_{\text{out}} \), i.e. for all \( \Xi \in \mathcal{X}_p \).

It is stressed that for \( \rho = 1 \) the requirement of \( \rho \)-stability of \( \hat{\Sigma} \) in Problems 1 and 2 coincides with its mere asymptotic stability.

3. **Main results**

In this section, Problems 1 and 2 are solved. In particular, formal results stating necessary and sufficient conditions under which strongly stabilizing compensators guaranteeing an infinite gain margin exist, as well as procedures to design such compensators, are given.

3.1. **Strong infinite gain margin \( \rho \)-stabilization for input perturbations**

**Theorem 1:** Under Assumption 1, Problem 1 is solvable if and only if

(i) system \( S \) is \( \rho \)-stabilizable and \( \rho \)-detectable;

(ii) each column of \( D \) has at least one non-zero entry.

**Remark 4:** Condition (ii) of Theorem 1 means that for each scalar unknown gain \( \xi \) on the main diagonal of \( \Xi \), the perturbed system \( S_{\Xi}^\text{in} \) has at least one non-zero scalar direct input/output connection affected by \( \xi \). Note that the necessity of condition (ii) of Theorem 1 when \( S \) is SISO (and then condition (ii) is equivalent to the condition \( D \neq 0 \)) was conjectured in Khargonekar et al. (1985). While the necessity of condition (i) of Theorem 1 is obvious, it is worth emphasizing that condition (ii) of Theorem 1 is necessary even if in Problem 1 the requirement of strong \( \rho \)-stabilization is weakened to mere \( \rho \)-stabilization (i.e. if the compensator \( \hat{K} \) is not required to be asymptotically stable; in fact the necessity proof of Theorem 1 does not rely on the stability of \( \hat{K} \)). Obviously, since every LTI compensator can be viewed as an LPTV compensator (with arbitrary period), condition (ii) of Theorem 1 is also necessary to achieve infinite gain margin \( \rho \)-stabilization by means of LTI compensators. In addition, it is stressed that conditions (i) and (ii) of Theorem 1 are still sufficient for the solvability of Problem 1 if Assumption 1 is violated, as can be easily seen.
Remark 5: Theorem 1, and in particular the necessity of its condition (ii), confirms a conjecture that was formulated in Khargonekar et al. (1985) about the necessity of a non-zero direct feed-through term in order to ensure infinite gain margin stabilization for a SISO plant. Moreover, Theorem 1 extends the results derived in Francis and Georgiou (1988) and Yan et al. (1994) for MIMO plants for when the same unknown scalar gain acts on all inputs; Theorem 1 solves the same kind of problem, but for when different unknown scalar gains act on every input, and an infinite gain margin (instead of an arbitrarily large one) has to be obtained, by means of a stable controller (thus obtaining a strong stabilization). In order to compare the results of this paper with those of Yan et al. (1994), note that system $S$ considered here could be the discrete-time model of a digitally controlled continuous-time system. With some minor modifications, the controller proposed here can solve the same problem studied in Francis and Georgiou (1988) and Yan et al. (1994), but under the weaker condition of only one non-zero entry in $D$, instead of condition (ii) of Theorem 1; however, condition (ii) of Theorem 1 is also weaker than one of the hypotheses of Lemma 4.1 in Yan et al. (1994) which requires $\text{rank}(D)$ to be greater than or equal to the maximum of the number of Jordan blocks associated with the unstable eigenvalues of $A$, whereas condition (ii) of Theorem 1 is satisfied even when $D$ has only a wholly non-zero row. In this framework, it is noted that, with some minor modifications, the controller proposed in Francis and Georgiou (1988) (see the proof of Theorem 3 there) could be designed under the slightly weaker hypotheses of Lemma 4.1 of Yan et al. (1994).†

Theorem 1 is proven next. Since the sufficiency proof is constructive (thus yielding a compensator solving Problem 1), but quite technical and lengthy, a qualitative explanation of the mechanism enabling the compensator to guarantee strong stabilization with infinite gain margin (Remark 6) and an algorithmic design procedure (Procedure 1) are separately given for the benefit of the reader not interested in the technicalities of the proof.

Proof of Theorem 1:

(Necessity) The necessity of condition (i) of Theorem 1 is obvious. Suppose, there exist a period $\omega \geq 1$ and an LPTV dynamic output feedback compensator $K_v$ of period $\omega$ that are able to solve Problem 1, but condition (ii) of Theorem 1 is not satisfied; then there is at least one column of $D_S^m$, say the $m$th, whose elements are all zero for all $\mathbf{z} \in X_p$. Define $\hat{S}$ as the closed-loop system composed by the feedback connection of $S^m$ and $K_v$.

Note that the well-posedness of the system is preserved when we open the loop at the $m$th scalar input of $S^m$. This act is equivalent to placing a zero gain on the $m$th scalar input of $S^m$, thus making the $m$th column of $D_S^m$ zero, while it is already zero by hypothesis. Therefore, if $\hat{S}$ is well-posed, this property is preserved under the aforementioned action. Then, suppose we open the loop at the $m$th scalar input of $S^m$ and we fix $\xi_v$ with $v = 1, \ldots, p$, $v \neq m$, to obtain the SISO LPTV system $\tilde{S}^m$. The associated LTI system $\tilde{S}^m_\omega$ has the matrix $D_S^{\omega}$ lower triangular with all zeros on the main diagonal. Therefore, closing the loop around $\tilde{S}^m_\omega$ (that is, coming back to the initial configuration of $\tilde{S}^m$), the closed-loop system $\hat{S}$ will be $\rho$-stable (that is, $\hat{S}^m_\omega$ will be $\rho^\omega$-stable) with an infinite gain margin for the variations of $\xi_m$, only if all the roots of the characteristic polynomial of $A_S^m + B_S^m (I_m - D_S^m)^{-1} C_S^m$ are smaller in modulus than $\rho^\omega$ for all $\xi_m \in [1, \infty)$. Notice that $B_S^m$ and $D_S^m$ have all the entries affected by the factor $\xi_m$, and that, as a consequence of the structure of $D_S^m$, $\det(I_m - D_S^m) = 1$, so that $(I_m - D_S^m)^{-1}$ is a matrix whose elements are polynomials in $\xi_m$. Denote with $p_{\xi_m}(\lambda)$ the characteristic polynomial of $A_S^m + B_S^m (I_m - D_S^m)^{-1} C_S^m$. Therefore $p_{\xi_m}(\lambda)$ is a monic polynomial in $\lambda$, whose coefficients of the powers of $\lambda$ with exponent less than $\bar{n}$ are polynomials in $\xi_m$; note that among these polynomials there must be at least a non-constant one as, by Assumption 1, if $\xi_m = 0$ the closed-loop system $\hat{S}$ is not $\rho$-stable, whereas for $\xi_m \geq 1$ it is $\rho$-stable, so that $p(\lambda) \neq p(\lambda)$. This, in turn, implies that $p_{\xi_m}(\lambda)$ has at least one coefficient whose modulus grows towards $\infty$ as $\xi_m \to +\infty$. This contradicts that all roots of $p_{\xi_m}(\lambda)$ are smaller than $\rho^\omega$ in modulus for all $\xi_m \in [1, \infty)$, since by hypothesis $\rho \leq 1$; in fact, if $p_{\xi_m}(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_n)$ and $|\lambda_i| < \rho^\omega$ for all $i = 1, \ldots, n$, and for all $\xi_m \in [1, \infty)$, then no coefficient of $p_{\xi_m}(\lambda)$ can have a modulus growing towards $\infty$ when $\xi_m \to +\infty$.

(Sufficiency) This part of the proof is carried out under the assumption that $S$ is reachable and observable (if this is not the case, by applying the same steps performed in this part of the proof to any minimal realization $S = (A, B, C, D)$ of $C(zI - A)^{-1}B + D$, a controller is obtained that will also work for $S$, provided that $S$ is $\rho$-stabilizable and $\rho$-detectable, since this implies that the eigenvalues of the non-reachable and/or unobservable parts of $S$ are smaller than $\rho$, in modulus). The structure of the sufficiency proof (which is constructive) will be as follows: (i) with reference to the control system depicted in figure 1, under the stated
hypotheses it will be described how to design the blocks composing the compensator (i.e. all blocks in figure 1 other than Σ and S); (ii) it will be shown that the designed compensator is asymptotically stable and the closed-loop system depicted in figure 1 is well-posed for all Σ ∈ Xp; (iii) it will be shown that the designed compensator succeeds in ρ-stabilizing every plant S2 ∈ S1.

(i): Define π = {1, ..., p} and χ = {1, ..., q}. Select c ∈ π, such that it is possible to partition π into c ordered subsets C1 = {s1,1, ..., s1,a1}, ..., Cc = {sc,1, ..., sc,ac}, \( \sum_{i=1}^{c} a_i = p \), such that, for every set Ci, \( i = 1, ..., c \), it is possible to find an ordered subset \( R_i = \{t_{i,1}, ..., t_{i,bi} \} \) of \( c \) such that matrix \( [D]_{(R_i,C_i)} \) is non-singular; note that the existence of such a \( c \) ∈ π is guaranteed by condition (ii) of Theorem 1, which implies that the choice c = p always satisfies the requirements (although this is not, in general, the most convenient choice, since it may give rise to a longer period \( \omega \) of the controller, as will be clear soon).

Let \( \eta = \{1, ..., n\} \). Set

\[
V = \sum_{j=0}^{c-1} A^j \left( B - (B^j)^{[\pi_\eta(\pi_\eta)\eta]} \right)
\]

and call \( v \) the observability index of system S. Recall that system S is reachable and observable, by assumption, and that there are infinitely many values of \( \sigma \in \mathbb{Z}^+ \) such that the number of distinct eigenvalues of \( A^\sigma \) is the same as the number of distinct eigenvalues of \( A \). Then, set \( \omega \) equal to the smallest value of such \( \sigma \) that is greater than or equal to \( c + v \) (so that \( \omega \geq 2 \)). Now, note that the reachability of \( (A,B) \) implies the controllability of \( (A^\omega, V) \), since for each left-side eigenvector \( \lambda \) of \( A \), and for each column \( b_a \) of \( B \) such that the relations

\[
\lambda = 0, \quad \lambda b_a \neq 0
\]

are satisfied, it is easy to see that, for every \( i \in \{0, ..., c - 1\} \), the relations

\[
v'(A^\omega - \lambda I) = 0, \quad v' A^\omega b_a \neq 0
\]

are also satisfied, thus implying the controllability of \( (A^\omega, V) \).

It is now needed to define \( \hat{L} \), \( \hat{R} \), \( \hat{Q} \) and \( \hat{W} \) in figure 1 as the \( \omega \)-periodic non-dynamic subcompensators, i.e. \( \omega \)-periodic gain matrices \( \hat{L}(k), \hat{R}(k), \hat{Q}(k) \) and \( \hat{W}(k) \) such that \( \hat{L}(h \omega + i) = \hat{L}_i, \hat{R}(h \omega + i) = \hat{R}_i, \hat{Q}(h \omega + i) = \hat{Q}_i \) and \( \hat{W}(h \omega + i) = \hat{W}_i \), for all \( h \) and for \( i = 0, ..., \omega - 1 \). Matrices \( L_i \in \mathbb{R}^{p \times p}, R_i \in \mathbb{R}^{p \times q}, Q_i \in \mathbb{R}^{p \times p} \) are defined as

\[
(L_{\omega-1})_{(c,C_i)} = \frac{1}{\omega} \sum_{k=0}^{\omega-1} L_i, \quad L_{\omega-1} = \frac{1}{\omega} \sum_{k=0}^{\omega-1} L_i
\]

\[
(R_{\omega-1})_{(c,R_i)} = \frac{1}{\omega} \sum_{k=0}^{\omega-1} R_i, \quad R_{\omega-1} = \frac{1}{\omega} \sum_{k=0}^{\omega-1} R_i
\]

\[
(Q_{\omega-1})_{(c,Q_i)} = \frac{1}{\omega} \sum_{k=0}^{\omega-1} Q_i, \quad Q_{\omega-1} = \frac{1}{\omega} \sum_{k=0}^{\omega-1} Q_i
\]

where the above choices of \( L_i \) and \( Q_i \) can always be made, and the same is true for the choice of \( R_i \) by virtue of the choice made for the sets \( C_i \) and \( R_i \).

Now matrices \( W_i \) will be defined. By virtue of the controllability of \( (A^\omega, V) \), compute \( K \in \mathbb{R}^{p \times n} \) such
that all the eigenvalues of $A^\omega + VK$ are equal to zero. Then, define
\[
U_2 := ([I_p + Q_{w-c}]R_{w-c} \cdots (I_p + Q_{w-1})R_{w-1}]
\]
\[
C_1 := \begin{bmatrix}
  C \\
  CA \\
  \vdots \\
  C A^{w-c-1}
\end{bmatrix}
\quad C_2 := \begin{bmatrix}
  CA^{w-c} \\
  \vdots \\
  C A^{w-1}
\end{bmatrix}
\]
(11)
In addition, taking into account that $C_1$ is full column-rank by the observability of $(A, C)$ and the relation $\omega - c \geq v$, define
\[
U_1 := (K - U_2 C_2) C_1^T
\]
(13)
Partition $U_1$ as $U_1 = [W_0 \ W_1 \ \cdots \ W_{w-1-c}]$, with $W_i \in \mathbb{R}^{p \times q}$, $i = 0, \ldots, \omega - 1 - c$, and set $W_i = 0_{p \times q}$, $i = \omega - c, \ldots, \omega - 1$.
Recalling that $\eta = \{1, \ldots, n\}$, compute $T \in \mathbb{R}^n$ such that $T(A^\omega + VK)T^{-1}$ is in Jordan form, and let $r$ be the degree of the minimal polynomial of $(A^\omega + VK)$. Fix $\gamma_1, \gamma_2, \ldots, \gamma_p \in \mathbb{R}^+$ such that $\gamma_i > 1$ for $i = 1, \ldots, p$, and
\[
\frac{1}{1 + \gamma_i} \| (TV)_{(s, j)} (KT^{-1})_{j, i} \|_\infty < \left(\frac{\rho^\omega + 1}{r} \right)^{1/\gamma_i} - 1
\]
(14)
Set $\Gamma = \text{diag}(\gamma_1, \ldots, \gamma_p)$.
Finally, let $\hat{H}$ be the LPTV dynamic system of period $\omega$ whose state space description is characterized by the matrices:
\[
A^\hat{H}(k) = 0_{p \times p} \quad B^\hat{H}(k) = I_p
\]
for $k = h\omega$
(15a)
\[
A^\hat{H}(k) = 0_{p \times p} \quad D^\hat{H}(k) = 0_{p \times p}
\]
(15b)
\[
A^\hat{H}(k) = I_p \quad B^\hat{H}(k) = I_p
\]
for $k = h\omega + i$, $i = 1, \ldots, \omega - 1$
\[
C^\hat{H}(k) = I_p \quad B^\hat{H}(k) = 0_{p \times p}
\]
(15b)
and define the overall compensator $\hat{K}$ as the connection of $\hat{W}$, $\hat{R}$, $\hat{Q}$, $\hat{H}$, $\hat{L}$ and $\Gamma$ according to the block diagram depicted in figure 1.
(ii): In order to see that the above designed compensator $\hat{K}$ is asymptotically stable, it is sufficient to notice that the only dynamic subcompensator, namely $\hat{H}$, has all its characteristic multipliers equal to zero, in view of (15a).
The well-posedness of the closed-loop system represented in figure 1 can be easily proven by considering that, as $\hat{H}$ is a strictly proper system, the only path along which well-posedness must be tested is that formed by the cascade connection of $\hat{L}$, $\Gamma$, $\hat{S}$ and $\hat{R}$. Moreover, as $\hat{L}(h\omega + i) = 0$ for $i = 0, \ldots, \omega - 1 - c$, well-posedness must be tested only at times $k = h\omega + i$ with $i = \omega - c, \ldots, \omega - 1$. Note that the product of the matrices $\hat{S}$ and $\Gamma$ in figure 1 is a matrix $Z = \text{diag}(\xi_1, \ldots, \xi_p)$ such that $Z \in \mathcal{X}_p$; matrix $Z$ will be often referred to in the sequel. By definition of the matrices involved, at times $k = h\omega + i$ with $i = \omega - c, \ldots, \omega - 1$ the relation
\[
R_D L_i = -L_i
\]
holds, and being $L_i$ and $Z$ diagonal matrices
\[
Z L_i = L_i Z
\]
therefore $\det (I - R_D Z L_i) = \det (I + L_i Z) \neq 0$, since all the diagonal entries of $Z$ are strictly greater than one; this proves well-posedness.
(iii): In order to prove the $\rho$-stability of system $\hat{S}$ in figure 1, note that the state $\hat{x}(k)$ of $\hat{S}$ is given by
\[
\hat{x}(k) = \begin{bmatrix}
  x^S(k) \\
  x^H(k)
\end{bmatrix}
\]
Denoting by $\hat{S}_0$ the associated system of $\hat{S}$ at time 0, the proof will be completed by showing that the following two equations hold
\[
x^S(h+1) = \begin{bmatrix}
  A^\omega + VZ(I + Z)^{-1}K & 0 \\
  * & 0
\end{bmatrix} x^S(h), \quad \forall h \geq 1
\]

\[
r(A^\omega + VZ(I + Z)^{-1}K) < \rho^\omega, \quad \forall \hat{S} \in \mathcal{X}_p
\]
where $r(\cdot)$ is the spectral radius of the argument matrix and the symbol $* \in (17a)$ stands for a submatrix whose value has no interest: in fact, by Proposition 1, equation (17) implies $\rho$-stability of $\hat{S}$ for all $\hat{S} \in \mathcal{X}_p$.
(iii.a): In order to prove (17a), note that, in view of (15a), the value of the state $\hat{x}^H(h\omega)$, for $\omega \geq 0$, will not influence the states of system $\hat{S}$ at all times $k > h\omega$, hence it will suffice to prove that $\hat{x}^S(h+1) = (A^\omega + VZ(I + Z)^{-1}K)\hat{x}^S(h)$. To this end, taking into account that $\hat{y}^L(h\omega + i) = 0$, for all $i = 0, \ldots, \omega - c - 1$, the following equation can be written
\[
\hat{x}^S(h+1) = A^\omega \hat{x}^S(h) + [A^{-1} B \cdots AB B] \times \begin{bmatrix}
  Z \hat{y}^L(h\omega + \omega - c) \\
  \vdots \\
  Z \hat{y}^L(h\omega + \omega - 1)
\end{bmatrix}
\]
Now, in order to compute the terms $\hat{y}^L(h\omega + i)$, $i = \omega - c, \ldots, \omega - 1$, remember that $\omega - c \geq 1$, so that,
for such values of \( i \)

\[
u^T(h_\omega + i) = x^H(h_\omega + i) + R_i y^S(h_\omega + i)
= x^H(h_\omega + i) + R_i \left( CA^i x^S_i(h)\right)
+ \sum_{j=\omega-c}^{i-1} CA^{i-1-j} B Z L_j u^T(h_\omega + j)\]

\[+ D Z L_j u^T(h_\omega + j)\]

and therefore, by the non-singularity of the matrix \( I - R_i D Z L_i \) for \( i = \omega - c, \ldots, \omega - 1 \)

\[
u^T(h_\omega + i) = (I - R_i D Z L_i)^{-1} \left( x^H(h_\omega + i) + R_i CA^i x^S_i(h)\right)
+ \sum_{j=\omega-c}^{i-1} CA^{i-1-j} B Z L_j u^T(h_\omega + j)\]

Hence, taking into account that, for \( i = \omega - c, \ldots, \omega - 1 \)

\[x^H(h_\omega + i) = x^H(h_\omega + \omega - c)
+ \sum_{j=0}^{i-1} Q_j R_j \left( CA^j x^S_j(h)\right)
+ \sum_{k=0}^{i-1} CA^{i-1-k} B Z L_k u^T(h_\omega + k)\]

\[+ D Z L_k u^T(h_\omega + j)\]

(20)

By using the relation

\[
\sum_{\sigma=0}^{\zeta} \sum_{\tau=0}^{\zeta} f(\sigma, \tau) = \sum_{\sigma=0}^{\zeta} \sum_{\tau=\sigma+1}^{\zeta} f(\tau, \sigma). \tag{22}
\]

Now, note that, in view of the fact that both \( Z \) and \( L_i \) are diagonal matrices, the vector \( \Delta(h, i) \) actually enters in the expression of \( y^k(h_\omega + i) \) pre-multiplied by \( L_i \). Note also that, in view of the special form of matrices \( L_i \), the following implication holds for all \( h \)

\[
\sum_{i=0}^{\omega-1} L_i \Delta(h, i) = 0 \implies L_i \Delta(h, i) = 0,
\]

for \( i = \omega - c, \ldots, \omega - 1 \). \tag{23}

By using the equation

\[
\left( \sum_{i=0}^{b} L_i \right) \sum_{j=0}^{d} L_j M(j) = \sum_{\max\{a,c\}}^{\min\{b,d\}} L_j M(j), \quad a, b, c, d \in \pi
\]

(which holds for any matrix or vector \( M(j) \) in view of the special form of matrices \( L_j \) and for any choice of the integers \( a, b, c, d \) in the set \( \pi = \{1, \ldots, p\} \) defined above), and taking into account (10), it can be easily seen that the term between square brackets after the last equality sign in (24) is null. Hence, in view of the implication (23), it follows that the term \( \Delta(h, i) \) can be neglected in (20). Consequently, taking into account (8b), (9b), (10c) and (15), by (20) \( y^k(h_\omega + i) \) can be written for \( i = \omega - c, \ldots, \omega - 1 \) as

\[
y^k(h_\omega + i) = L_i (I + Z L_i)^{-1} \left( x^H(h_\omega + \omega - c)
+ \sum_{j=0}^{i-1} Q_j R_j CA^j \right) x^S_i(h)\]

\[+ \sum_{j=\omega-c}^{i-1} Q_j R_j CA^{i-1-j} B Z L_j u^T(h_\omega + j)\]

\[\times Z L_j u^T(h_\omega + j)\] \tag{25}

Note that, by (7), the equality \( V = \sum_{i=0}^{c-1} A^i B L_{\omega - 1 - i} \) holds, since \( B - (B^j)^{\{0,c\}} = BL_{\omega - j} \), for \( j = 1, \ldots, c \). Hence, starting from (18), taking into account the
special form of matrices $L_{a_{i-1},i}$, for $i = 0, \ldots, c - 1$, and $Z$, the definitions of $U_1$, $U_2$, $C_1$, $C_2$ in (11), (12) and (13), the definition of $W_j$, the identities

$$
L_iR_j = R_j, \quad \text{for } i = \omega - c, \ldots, \omega - 1
$$

$$
L_iL_j = L_j, \quad \text{for } i = \omega - c, \ldots, \omega - 1
$$

$$
Q_{a_{i-1},i} = \left( \sum_{j=0}^{c-2-i} L_{a_{i-1},j} \right) Q_{a_{i-1},i}, \quad \text{for } i = 0, \ldots, c - 2
$$

$$
L_iL_j = 0, \quad \text{for } i \neq j
$$

and (8b), (9b) and (10), the following computations can be carried out

$$
\sigma^N(b + 1) = \left( A^w + \sum_{i=0}^{c-1} A^w B_{a_{i-1},i} Z(I + Z)^{-1}
\times \left( \sum_{j=0}^{c-2-i} W_j C_{A^w}^i + R_{a_{i-1},c} A^w \right) + \frac{\sum_{j=0}^{c-2-i} Q_j R_j C_{A^w}^j}{}
\right) \sigma^N(h)
$$

$$
= \left( A^w + VZ(I + Z)^{-1} U_1 C_1 + \sum_{i=0}^{c-1} A^w B_{a_{i-1},i} Z(I + Z)^{-1}
\times \left( L_{a_{i-1},i} R_{a_{i-1},c} A^w \right) + \sum_{j=0}^{c-2-i} Q_j R_j C_{A^w}^j
\right) \sigma^N(h)
$$

$$
= \left( A^w + VZ(I + Z)^{-1} U_1 C_1 + \sum_{i=0}^{c-1} L_{a_{i-1},i} R_{a_{i-1},c} A^w \right)
\times \left( L_{a_{i-1},i} R_{a_{i-1},c} A^w \right) + \sum_{j=0}^{c-2-i} Q_j R_j C_{A^w}^j
\right) \sigma^N(h)
$$

$$
\sigma^N(h) = \left( A^w + VZ(I + Z)^{-1} U_1 C_1 + \sum_{i=0}^{c-1} L_{a_{i-1},i} R_{a_{i-1},c} A^w \right)
\times \left( \sum_{j=0}^{c-2-i} Q_j R_j C_{A^w}^j \right) \sigma^N(h)
$$

$$
\text{(26)}
$$

It is then easy to conclude that (17a) holds, taking into account that $K = U_1 C_1 + U_2 C_2$ in view of (13).

(iii.b): As for (17b), note that $r(A^w + VZ(I + Z)^{-1}) < \rho^\alpha$, and only if, for any $\tau \in \mathbb{Z}$, $r((A^w + VZ(I + Z)^{-1})^\tau) < \rho^\alpha$, and that, as for $G \in \mathbb{R}^{g \times f}$, $H \in \mathbb{R}^{f \times h}$, $s := [1, 2, \ldots, g]$ and $t := [1, 2, \ldots, h]$, $G H = \sum_{i=1}^{g} (G)_{(i,s)} (H)_{(t,i)}$, then the following expansion is possible for any non-singular $T \in \mathbb{R}^{n \times n}$

$$
TV(I + Z)^{-1} KT^{-1} = \sum_{i=1}^{n} \left( \frac{1}{1 + \xi_i} \langle TV \rangle_{(i,s)} (KT_1)^{-1}(i,t) \right)
$$

$$
(27)
$$

where $\eta = \{1, \ldots, \eta\}$. Now, let $T$ be such that $T(A^w + VK)T^{-1}$ is in Jordan form, and $\tau$ be the degree of the minimal polynomial of $(A^w + VK)$, as set right before (14). Since $K$ has been chosen such that all the eigenvalues of $(A^w + VK)$ are equal to zero, in the special case when $\tau = 1$ it is $A^w + VK = 0$; in this case note that

$$
A^w + VZ(I + Z)^{-1} K = A^w + VK - V(I + Z)^{-1} K = -V(I + Z)^{-1} K
$$

so that (14), (27), (28) and the relation

$$
\xi_i = \gamma_i \delta_i \geq \gamma_i > 1
$$

implies that

$$
\rho^\alpha
$$

testifying T \in \mathbb{R}^{n \times n}

$$
\text{of the form } (I + Z)^{-1} KT^{-1} \leq \sum_{i=1}^{\rho^\alpha} \left( \frac{1}{1 + \xi_i} \frac{\|TV\|_{(i,s)} (KT_1)^{-1}(i,t)}{1 + \xi_i} \right)
$$

$$
\text{with } \rho^\alpha
$$

thus proving (17b) for $\tau = 1$. If, on the contrary, $\tau > 1$, note further that, if $M$, $N \in \mathbb{R}^{n \times n}$, then $(M + N)^\gamma = M^\gamma + R(M, N, \gamma)$, where

$$
\|R(M, N, \gamma)\| \leq \sum_{h=0}^{\tau-1} \left( \frac{\tau}{h} \right) \|M\|_\infty \|N\|_\infty \|T\|_h
$$

$$
= \left( \|M\|_\infty + \|N\|_\infty \right)^\tau - \|M\|_\infty^n.
$$

(30)

Define

$$
M = T(A^w + VK)T^{-1},
$$

$$
N = -TV(I + Z)^{-1} KT^{-1}.
$$

The definitions of $T$ and $M$ imply that $\|M\|_\infty = 1$ and $\|M^\gamma\|_\infty = 0$, so that by (14), (27), (29) and (30) the following relations hold

$$
r((A^w + VZ(I + Z)^{-1} KT^{-1})^\gamma
$$

$$
\leq \|T(A^w + VZ(I + Z)^{-1} KT^{-1})^\gamma\|_\infty
$$

$$
= \|T(A^w + VK)T^{-1} - T(V(I + Z)^{-1} K)T^{-1}\|_\infty
$$

$$
= \|M + N\|_\infty
$$

$$
\leq \|M\|_\infty^n + |R(M, N, \gamma)|\|N\|_\infty^n
$$

$$
\leq 0 + (\|M\|_\infty^n + \|N\|_\infty^n)^\tau - \|M\|_\infty^n
$$

$$
= \left( 1 + \sum_{i=1}^{n} \frac{1}{1 + \xi_i} \frac{(TV)_{(i,s)} (KT_1)^{-1}(i,t)}{1 + \xi_i} \right)^n - 1
$$

$$
\leq 1 + \left( \frac{\rho^\alpha + 1}{\rho^\alpha} \right)^\tau - 1
$$

$$
= \rho^\alpha
$$

(31)

thus proving (17b) for $\tau > 1$. This completes the proof of the $\rho$-stability of system $\Sigma$ for every plant $S_{\eta}^\infty \in S_{\eta}^\infty$.

\textbf{Remark 6: } The control system in figure 1 is an extension of a simpler LPTV control scheme which makes use of an asymptotically stable dynamic output
feedback compensator to stabilize an exactly known (i.e. $\Xi = I$) reachable and observable LTI plant $S$. In such a simpler control scheme (based on ideas inspired by Chamas and Leondes (1978) and Haghiwara and Araki (1988)), the blocks $\Gamma$, $R$ and $Q$ are absent (i.e. $\Gamma = I$, $R \equiv 0$ and $Q \equiv 0$) and the strictly proper LPTV controller is designed as follows. Let $\nu [\mu]$ be the observability [reachability] index of $S$, and choose the period $\omega := \nu + \mu$. Let $V \in \mathbb{R}^{n \times n}$ and $K \in \mathbb{R}^{n \times n}$ be any two matrices such that all the eigenvalues of $A^\nu + VK$ are zero. By observability and reachability of $S$, the equations $K' = [C' (CA')^{-1} (CA')^{-1}]^T$ and $V = A^{-1}B \cdots AB BM$ have solutions $N \in \mathbb{R}^{n \times n}$ and $M \in \mathbb{R}^{n \times n}$. Define the periodic gains $\bar{W}(k) \in \mathbb{R}^{n \times n}$ and $\bar{L}(k) \in \mathbb{R}^{n \times n}$ as

$$
\bar{W}(k) = \begin{cases} 
(N)_{(1, \ldots, s), [i, q+1, \ldots, q+q]} & \text{for } k = \omega h + i, \quad i = 0, \ldots, \nu - 1 \\
0 & \text{for } k = \omega h + i, \quad i = \nu, \ldots, \omega - 1 \\
0 & \text{for } k = \omega h + i, \quad i = 0, \ldots, \nu - 1 \\
(M)_{((i, q+1, \ldots, i+q+1), \ldots, (1, \ldots, i))} & \text{for } k = \omega h + i, \quad i = \nu, \ldots, \omega - 1
\end{cases}
$$

and $\bar{\bar{L}}(k)$ as in equations (15) after replacing $p$ by $s$. During the first part of the period (estimation interval), the plant $S$ is in free motion ($\bar{L}(k) = 0$) and the state of $\bar{\bar{H}}$ at time $k = \omega h + v$ is brought to the value $K_x \Xi(h) = \sum_{i=0}^{v} \bar{W}(\omega h + i) CA' \Xi(h)$; during the second part of the period (control interval), the state (and then the output) of $\bar{\bar{H}}$ is held constant ($\bar{W}(k) = 0$) and the plant is driven through $\bar{L}$, so that $u^\nu(\omega h + i) = \bar{L}(\omega h + i) K_x \Xi(h)$ for $\nu = \omega - \mu, \ldots, \omega - 1$, and thus $x^\nu(h + 1) = A^\nu x^\nu(h) + \sum_{i=0}^{\nu-1} A^{\nu-i-1} B \bar{L}(\omega h + v + i) x^\nu(h) = (A^\nu + VK)x^\nu(h)$ and the closed-loop characteristic multipliers are all zero.

Clearly, already in the presence of an input unbounded multiplicative uncertainty of the type $\Xi = \xi I$ the just-described controller fails stabilizing the plant, since in this case the obtained closed-loop monodromy matrix is $A^\nu + VK$, which under Assumption 1 has at least one unbounded eigenvalue for $\xi \to \infty$ (as a matter of fact, a proof similar to the final part of the necessity proof of Theorem 1 shows that such an unbounded eigenvalue appears for any strictly proper compensator in the presence of such unbounded multiplicative uncertainties). However, $\rho$-stabilization can be obtained by introducing the additional blocks shown in Figure 1, which create suitable direct feedthrough paths (making the controller not strictly proper).

Note that, since the uncertainty affects just the input to the plant, the direct feedthrough is only needed during the ‘control interval’ (see Steps 1.3 and 1.4 of the subsequent Procedure 1); however, since (for reasons that will be clarified later) the state of $\bar{\bar{H}}$ changes during the ‘control interval’ (when $\bar{W} = 0$ but $\bar{Q} \neq 0$, see Steps 1.5 and 1.8 of Procedure 1), the function of $x^\nu(h)$ to be stored as the state of $\bar{\bar{H}}$ during the ‘estimation interval’ is not simply $K_x \Xi(h)$ (a choice guaranteeing $u^\nu(\omega h + i) = \bar{L}(\omega h + i) K_x \Xi(h)$ for $\nu = \omega - \mu, \ldots, \omega - 1$ in the above-mentioned simpler scheme), but must be suitably modified (see Step 1.6 of Procedure 1 and relation (19)) in order to guarantee

$$
u^\nu(\omega h + i) = \xi^\nu \bar{L}(\omega h + i) \left( y^\nu(\omega h + i) + y^H(\omega h + i) \right)
$$

for $i = \omega - c, \ldots, \omega - 1$ (by (25) and the choice of $W_j$ in Step 1.6 of Procedure 1), and then that, for sufficiently large $\gamma$, $u^\nu(\omega h + i) \approx \bar{L}(\omega h + i) K_x \Xi(h)$ for $i = \omega - c, \ldots, \omega - 1$, as it will be now clarified.

In fact, considering for simplicity the above-mentioned special type of input uncertainty $\Xi = \xi I$, $\xi \in [1, +\infty)$, specialize the choice of $\Gamma$ (see Step 1.7 of Procedure 1) by setting $\Gamma = y \Gamma$ with $\gamma = \max\{y_1, \ldots, y_p\}$. By introducing $\bar{\bar{R}}$ and $\bar{\bar{L}}$ according to Procedure 1, the series connection of $\bar{L}$, $\Gamma$, $\Xi$, $S$ and $R$ is a periodic system with diagonal feedthrough terms with the only non-zero elements all equal to $-\gamma$. Making reference to the sufficiency proof of Theorem 1, it is then possible to show that the dependence of $u^\nu(\omega h + i) = \gamma^\nu \bar{L}(\omega h + i)$ on $u^\nu(\omega h + j)$ for $j \in [0, i-1]$ (such dependence is expressed by means of the term $\Delta(h, i)$ in (20), depending on $ZL, \mu^\nu(\omega h + j)$ which is equal to $u^\nu(\omega h + j)$ can be nulled by a suitable choice of $\bar{Q}$ (see Step 1.5 of Procedure 1 and equation (21)), so that $u^\nu(\omega h + i)$ can be simply expressed as a function of $x^\nu(h)$ (see (25)). The price paid to achieve this simplification is that the state of $\bar{\bar{H}}$ is no longer constant during the control interval, and then to achieve the same overall effect on $x^\nu(h + 1)$ that was achieved in the simpler control scheme by having $x^\nu(\omega h + i) = K_x \Xi(h)$ for $i = \omega - \mu, \ldots, \omega - 1$ it is necessary to have $x^\nu(\omega h + \omega - c)$ at a suitable value, different from $K_x \Xi(h)$, i.e. it is required to modify (with respect to the simpler control scheme) the design equation for $\bar{W}$ for $i = \omega, \ldots, \omega - c - 1$ according to Step 1.6 of Procedure 1. Finally, the resulting closed-loop monodromy matrix

$$A^\nu + \frac{\xi \gamma}{1 + \xi \gamma} VK = (A^\nu + VK) - \frac{1}{1 + \xi \gamma} VK
$$

is obtained, where the term $(1/(1 + \xi \gamma))VK$ can be seen as a perturbation from the desired behaviour dictated by $A^\nu + VK$. Clearly, since $\lim_{\gamma \to +\infty}(1/(1 + \xi \gamma))VK = 0$, by continuity choosing a sufficiently high value for $\gamma$ (as in Step 1.7 of Procedure 1), the closed-loop characteristic multipliers can be guaranteed to be arbitrarily close to the eigenvalues of $A^\nu + VK$, for all allowable values.
of $\xi$, and then robust stabilization follows. Note that, contrary to what happens for the above-mentioned simpler LPTV compensator, a higher value of $\xi$ renders the closed-loop system behaviour more and more similar to the desired closed-loop behaviour given by $A^\omega + VK$ (in particular, as $\xi \to \infty$, the closed-loop system tends to become deadbeat stable).

The following design procedure of a solution to Problem 1 will refer to figure 1, where the structure of the proposed control system is depicted.

Procedure 1: (Valid under the hypotheses that system $S$ is reachable and observable and condition (ii) of Theorem 1 holds) Design of a solution to Problem 1.

Step 1.1: (Fix the integer $\epsilon$ and the sets of integers $R_i$ and $C_i$ for $i = 1, \ldots, c$.) Define $\pi = \{1, \ldots, p\}$ and $\chi = \{1, \ldots, q\}$. Select $\epsilon \in \pi$, such that it is possible to partition $\pi$ into $\epsilon$ ordered subsets $C_i = \{s_i, 1, \ldots, s_i, a_i\}$, $\epsilon = \sum_{i=1}^{\epsilon} \alpha_i = p$, such that, for every set $C_i$, $i = 1, \ldots, c$, it is possible to find an ordered subset $R_i = \{t_i, 1, \ldots, t_i, a_i\}$ of $\chi$ such that matrix $(D(\Omega_c, \Omega_j))$ is non-singular.

Step 1.2: (Choice of the period $\omega$.) Let $\eta = \{1, \ldots, n\}$. Let $V$ be given by (7), and call $v$ the observability index of system $S$. Set $\omega \in \mathbb{Z}^+$ equal to the smallest integer satisfying $\omega \geq v + c$ such that the pair $(A^\omega, V)$ is controllable.

Step 1.3: (Fix matrices $L_{\epsilon-i}$ for $i = 1, \ldots, \omega$.) For $i = 1, \ldots, \omega$, define $L_{\epsilon-i} \in \mathbb{R}^{p \times \pi}$ as in (8a) and (8b).

Step 1.4: (Fix matrices $R_{\epsilon-i}$ for $i = 1, \ldots, \omega$.) For $i = 1, \ldots, \omega$, define $R_{\epsilon-i} \in \mathbb{R}^{\pi \times q}$ as in (9a) and (9b).

Step 1.5: (Fix matrices $Q_{\epsilon-i}$ for $i = 1, \ldots, \omega$.) For $i = 1, \ldots, \omega$, recursively define $Q_{\epsilon-i} \in \mathbb{R}^{\pi \times q}$ as in (10a), (10b) and (10c).

Step 1.6: (Fix matrices $W_i$ for $i = 0, \ldots, \omega - 1$.)

Compute $K \in \mathbb{R}^{p \times q}$ such that all the eigenvalues of $A^\omega + VK$ are equal to zero. Then, define $U_2$, $C_1$, $C_2$ as in (11) and (12), and compute $U_1$ as in (13). Partition $U_1$ as $U_1 = [W_0 \ W_1 \ \cdots \ W_{\omega-1-c}]$, with $W_i \in \mathbb{R}^{p \times q}$, $i = 0, \ldots, \omega - 1 - c$, and set $W_i = 0_{p \times q}$ for $i = \omega - c, \ldots, \omega - 1$.

Step 1.7: (Choice of the matrix $\Gamma$.) As in Step 1.2, let $\eta = \{1, \ldots, n\}$. Compute $T \in \mathbb{R}^{\pi \times \eta}$ such that $T(A^\omega + VK)T^{-1}$ is in Jordan form, and let $r$ be the degree of the minimal polynomial of $(A^\omega + VK)$. Fix $\gamma_1, \ldots, \gamma_p \in \mathbb{R}^+$ such that $\gamma_i > 1$ for $i = 1, \ldots, p$, and (14) is satisfied. Set $\Gamma = \text{diag}(\gamma_1, \ldots, \gamma_p)$.

Step 1.8: (Fix subcompensators $\tilde{L}$, $\tilde{R}$, $\tilde{Q}$, $\tilde{W}$ and $\tilde{H}$.)

Define $L$, $R$, $Q$ and $W$ as the $\omega$-periodic non-dynamic gain matrices $L(k)$, $R(k)$, $Q(k)$ and $W(k)$ such that $L(\omega h + i) = L_i$, $R(\omega h + i) = R_i$, $Q(\omega h + i) = Q_i$, and $W(\omega h + i) = W_i$, for all $h$ and for $i = 0, \ldots, \omega - 1$. Furthermore, set $\tilde{H}$ as the LPTV dynamic system whose state space description is characterized by matrices (15).

Step 1.9: Define the overall compensator $\tilde{K}$ as the connection of $W$, $R$, $Q$, $H$, $L$ and $\Gamma$ according to the block diagram depicted in figure 1.

Remark 7: If, at Step 1.7 of Procedure 1, $\gamma_1, \ldots, \gamma_p \in \mathbb{R}^+$ are fixed such that $\gamma_i > 1$ for $i = 1, \ldots, p$, and the following relation is satisfied

$$\frac{1}{\gamma_i - 1} \left\| \left( (T^{(\pi, i)})(K^{T^{-1}})_{i, n} \right)_{i, n} \right\| \leq \frac{(\rho^{\omega} + 1)_{i, n} - 1}{\rho}$$

(where $\eta = \{1, \ldots, n\}$, instead of (14), then the control system in figure 1 satisfies all the requirements of Problem 1 for the whole family of perturbations defined by $\mathcal{T}_p := \{\mathcal{E} \in \mathbb{R}^{p \times p} : \mathcal{E} = \text{diag}(\xi_1, \ldots, \xi_p), \xi_i \in (-\infty, -1] \cup [1, +\infty), i = 1, \ldots, p\}$.

3.2. Strong infinite gain margin $\rho$-stabilization for output perturbations

Theorem 2: Under Assumption 2, Problem 2 is solvable if and only if

(i) system $S$ is $\rho$-stabilizable and $\rho$-detectable;

(ii) each row of $D$ has at least one non-zero entry.

Remark 8: Clearly, Theorem 1 and Theorem 2 are completely dual. For brevity, remarks and comments given for the case of input multiplicative perturbations (i.e., with respect to Problem 1 and Theorem 1) are not repeated, taking for granted that completely dual considerations hold for the case of output multiplicative perturbations (i.e., with respect to Problem 2 and Theorem 2). However, taking into account that duality for time-varying systems requires careful handling of time-dependent quantities, both a proof of Theorem 2 by duality and a procedure for the synthesis of a compensator solving Problem 2 will be detailed next.

The following design procedure of a solution to Problem 2 will refer to figure 2, where the structure of the proposed control system is depicted.

 Procedure 2: (Valid under the hypotheses that system $S$ is reachable and observable and condition (ii) of Theorem 2 holds.) Design of a solution to Problem 2.

\[\text{\textsuperscript{\dagger}}\text{If this is not the case, see note 2 to Procedure 1.}\]
Step 2.1: (Fix the integer $c$ and the sets of integers $C_i$ and $R_i$ for $i = 0, \ldots, c - 1$.) Define $\pi = \{1, \ldots, p\}$ and $\chi = \{1, \ldots, q\}$. Select $c \in \chi$, such that it is possible to partition $\chi$ into $c$ ordered subsets $C_0 = \{s_0, a_0\}, \ldots, C_{c-1} = \{s_{c-1}, 1, \ldots, a_{c-1}\}$, $\sum_{i=0}^{c-1} a_i = q$, such that, for every set $C_i$, $i = 0, \ldots, c - 1$, it is possible to find an ordered subset $R_i = \{l_{i,1}, \ldots, l_{i,a_i}\}$ of $\pi$ such that matrix $\{D_{i}(C_i, R_i)\}$ is non-singular.

Step 2.2: (Choice of the period $\omega$.) Let $\eta = \{1, \ldots, n\}$. Set $V = \sum_{j=0}^{c-1} (C - (C_i)^{\eta, \eta}) A^j$, and call $\nu$ the reachability index of system $S$. Set $\omega \in \mathbb{Z}^+$ equal to the smallest integer satisfying $\omega \geq v + c$ such that the pair $(A^\omega, V)$ is reconstructible.

Step 2.3: (Fix matrices $L_i$ for $i = 0, \ldots, \omega - 1$.) Define $L_i \in \mathbb{R}^{p \times q}$ as

\[
\langle L_i \rangle_{(C_i, C_i)} = I_{a_i}, \quad \langle L_i \rangle_{(C_i, C_j)} = 0, \quad i = 0, \ldots, c - 1
\]

\[
L_i = 0_{p \times q}, \quad i = c, \ldots, \omega - 1.
\]

Step 2.4: (Fix matrices $R_i$ for $i = 0, \ldots, \omega - 1$.) Define $R_i \in \mathbb{R}^{p \times q}$ as

\[
\langle R_i \rangle_{(R_i, C_i)} = -((D_{i})_{(C_i, R_i)})^{-1}, \quad \langle R_i \rangle_{(R_i, C_j)} = 0, \quad i = 0, \ldots, c - 1
\]

\[
R_i = 0_{p \times q}, \quad i = c, \ldots, \omega - 1.
\]

Step 2.5: (Fix matrices $Q_i$ for $i = 0, \ldots, \omega - 1$.) Recursively define $Q_i \in \mathbb{R}^{p \times q}$ as

\[
Q_0 := 0
\]

\[
Q_i := \sum_{j=0}^{i-1} C A^{i-j} B R_j + \sum_{k=i+1}^{i-1} C A^{i-k} B R_k Q_k L_j, \quad i = 1, \ldots, \omega - 1
\]

\[
Q_i := 0_{p \times q}, \quad i = c, \ldots, \omega - 1.
\]

Step 2.6: (Fix matrices $W_i$ for $i = 0, \ldots, \omega - 1$.) Set $W_i = 0_{p \times q}, \quad i = 0, \ldots, c - 1$. Compute $K \in \mathbb{R}^{p \times q}$ such that all the eigenvalues of $A^\omega + KV$ are equal to zero. Then, define

\[
U_2 := \\
= \begin{bmatrix} R_{c-1}(I_q + Q_{c-1}) \\
\vdots \\
R_0(I_q + Q_0) \\
\end{bmatrix}
\]

\[
B_1 = \begin{bmatrix} B & AB & \cdots & A^{\omega-1}B \\
\end{bmatrix} \quad \text{and} \\
B_2 = \begin{bmatrix} A^{\omega-1}B & \cdots & A^{\omega-1}B \\
\end{bmatrix}
\]

and compute $U_1 := B_1^T(K - B_2 U_2)$. Partition $U_1$ as

\[
U_1 = \begin{bmatrix} W_{\omega-1} \\
\vdots \\
W_c \\
\end{bmatrix}
\]

with $W_i \in \mathbb{R}^{p \times q}, \ i = c, \ldots, \omega - 1$.

Step 2.7: (Choice of the matrix $\Gamma$.) As in Step 2.2, let $\eta = \{1, \ldots, n\}$. Compute $T \in \mathbb{R}^{p \times q}$ such that $T((A^\omega)^{\tau} + V'K)T^{-1}$ is in Jordan form, and let $\tau$ be the degree of the minimal polynomial of $(A^\omega + KV)$. Fix $\gamma_1, \gamma_2, \gamma_q \in \mathbb{R}^+$ such that $\gamma_i > 1$ for $i = 1, \ldots, q$, and

\[
\frac{1}{1 + \gamma_i} \|TV'^{(\eta, \eta)}(K'T^{-1})_{(i, i)}\|_{\infty} < \left(\frac{\rho^\omega + 1}{\tau} - 1\right)^{-1}. \quad (32)
\]

Set $\Gamma = \text{diag}(\gamma_1, \ldots, \gamma_q)$.

Step 2.8: (Fix subcompensators $\hat{L}, \hat{R}, \hat{Q}, \hat{W}$ and $\hat{H}$.) Define $\hat{L}, \hat{R}, \hat{Q}$ and $\hat{W}$ as $\omega$-periodic non-dynamic gain matrices $L(k), R(k), Q(k)$ and $W(k)$ such that $L(\omega h + i) = L_i, R(\omega h + i) = R_i, Q(\omega h + i) = Q_i$ and $W(\omega h + i) = W_i$, for all $h$ and for $i = 0, \ldots, \omega - 1$. Furthermore, set $\hat{H}$ as the LPTV dynamic system.
whose state space description is characterized by the matrices

\[
\begin{align*}
A^H(k) &= I_q & B^H(k) &= I_q & C^H(k) &= I_q & D^H(k) &= 0_{q \times q} \\
&\text{for } k = h\omega + i, & i = 0, \ldots, \omega - 2 \tag{33a}
\end{align*}
\]

\[
\begin{align*}
A^H(k) &= 0_{q \times q} & B^H(k) &= 0_{q \times q} & C^H(k) &= I_q & D^H(k) &= 0_{q \times q} \\
&\text{for } k = h\omega + \omega - 1. \tag{33b}
\end{align*}
\]

Step 2.9: Define the overall compensator \( \tilde{K} \) as the connection of \( \Gamma, \tilde{L}, \tilde{H}, \tilde{Q}, \tilde{R} \) and \( \tilde{W} \) according to the block diagram depicted in figure 2.

In order to prove by duality Theorem 2, the relevant definitions and properties (Weiss 1972, Grasselli and Longhi 1987) are recalled next.

For a given \( k_0 \in \mathbb{Z} \), the dual system \( \tilde{\Theta}^{d,k_0} \) of an LPTV system \( \Theta = (A(k), B(k), C(k), D(k)) \) around \( k_0 \) is defined as

\[
\tilde{\Theta}^{d,k_0} = (A^d(k), B^d(k), C^d(k), D^d(k)) \text{ where } \forall \tilde{n} \in \mathbb{Z}
\]

\[
\begin{align*}
\tilde{A}^d(k_0 + \tilde{n}) &= \tilde{A}(k_0 - \tilde{n} - 1) \\
\tilde{B}^d(k_0 + \tilde{n}) &= \tilde{C}(k_0 - \tilde{n} - 1) \\
\tilde{C}^d(k_0 + \tilde{n}) &= \tilde{B}(k_0 - \tilde{n} - 1) \\
\tilde{D}^d(k_0 + \tilde{n}) &= \tilde{D}(k_0 - \tilde{n} - 1).
\end{align*}
\]

As expected, the dual system of \( \tilde{\Theta}^{d,k_0} \) around \( k_0 \) is again \( \tilde{\Theta} \); hence, being the dual of each other, they can be referred to as a dual pair. Due to the time-singularity between the matrices describing \( \Theta \) and its dual \( \tilde{\Theta}^{d,k_0} \), the latter is also \( \omega \)-periodic and is straightforward to see that the state transition matrix \( \tilde{\Phi}^d(\cdot) \) of \( \tilde{\Theta}^{d,k_0} \) is related to the state transition matrix \( \Phi^d(\cdot) \) of \( \Theta \) by the relation

\[
\tilde{\Phi}^d(k_0 + h_1, k_0 + h_0) = \tilde{\Phi}(k_0 - h_0, k_0 - h_1),
\]

\[
h_1 > h_0, \quad h_1, h_0 \in \mathbb{Z}. \tag{34}
\]

By (34), it is easy to see that \( \tilde{\Theta}^{d,k_0} \) is \( \rho \)-stable if and only if \( \Theta \) is \( \rho \)-stable, since \( \tilde{\Theta} \) and \( \tilde{\Theta}^{d,k_0} \) have the same monodromy matrix within a transposition, and hence they have the same characteristic multipliers.

It is straightforward to see that, given any two LPTV systems \( \Theta_1 \) and \( \Theta_2 \) having the same period \( \omega \), the following properties hold:

- the dual system around \( k_0 \) of the series connection of \( \Theta_1 \) followed by \( \Theta_2 \) is the series connection of \( \tilde{\Theta}_1^{d,k_0} \) followed by \( \tilde{\Theta}_2^{d,k_0} \);
- the dual system around \( k_0 \) of the parallel connection of \( \Theta_1 \) and \( \Theta_2 \) is the parallel connection of \( \tilde{\Theta}_1^{d,k_0} \) and \( \tilde{\Theta}_2^{d,k_0} \);
- the dual system around \( k_0 \) of the feedback connection of \( \tilde{\Theta}_1 \) feedback by \( \tilde{\Theta}_2 \) is the feedback connection of \( \tilde{\Theta}_1^{d,k_0} \) feedback by \( \tilde{\Theta}_2^{d,k_0} \), which is well posed if and only if the feedback connection of \( \tilde{\Theta}_1 \) feedback by \( \tilde{\Theta}_2 \) is well posed.

In the following, for simplicity of notation the dual system \( \tilde{\Theta}^{0,0} \) of an LPTV system \( \Theta \) around \( k_0 = 0 \) will be denoted simply by \( \tilde{\Theta}^d \) and will be called simply the dual system of \( \Theta \).

**Proof of Theorem 2:**

*(Necessity)* The necessity of condition (i) of Theorem 2 is obvious. The necessity of condition (ii) of Theorem 2 follows by duality from the necessity of condition (ii) of Theorem 1 and the relation \( D^d = D' \).

*(Sufficiency)* To prove that Procedure 2 can be completed and the compensator designed using Procedure 2 is asymptotically stable, similar steps as in the proof of Theorem 1 can be followed. The proof that the closed-loop system thus obtained and depicted in figure 2 is well posed and \( \rho \)-stable for all \( \Xi \in \mathcal{X}_q \), can be based directly on duality as follows. Apply Procedure 1 to the dual system \( \tilde{\Sigma}^d \) of \( \Sigma \) and call \( \tilde{W}_\alpha, \tilde{R}_\alpha, \tilde{Q}_\alpha, \tilde{H}_\alpha, \tilde{L}_\alpha \) and \( \Gamma_\alpha \), respectively, the subcompensators thus obtained, \( \tilde{\Theta}_\alpha \) the resulting overall compensator, \( \Xi \in \mathcal{X}_q \), as previously defined, the relevant admissible input multiplicative perturbation to be considered within Problem 1 for \( \Sigma^d \), and \( \tilde{\Sigma}_\alpha \) the resulting control system, which is depicted in figure 3(a). Since \( \Sigma^d \) actually satisfies the same hypotheses that are satisfied by system \( \Sigma \) when Procedure 1 is applied to \( \Sigma \) as in the sufficiency proof of Theorem 1, system \( \tilde{\Sigma}_\alpha \) is well posed and \( \rho \)-stable for all \( \Xi \in \mathcal{X}_q \). Therefore the dual system \( \tilde{\Sigma}^d \) of system \( \tilde{\Sigma}_\alpha \), which is made up of the dual connections of the dual subsystems \( \tilde{W}_\alpha^d, \tilde{R}_\alpha^d, \tilde{Q}_\alpha^d, \tilde{H}_\alpha^d, \tilde{L}_\alpha^d, \Gamma_\alpha^d \), \( \Xi^d \) and \( \tilde{\Sigma}^d \) (according to the statements just before this proof), and is depicted in figure 3(b), is well posed and \( \rho \)-stable for all \( \Xi \in \mathcal{X}_q \). Now it can be easily checked that

- (i) the subsystems in figure 3(b) satisfy the relations \( \Xi^d = \Xi, \Gamma^d = \Gamma, W^d = W, R^d = R, Q^d = Q, H^d = H, L^d = L \), where \( \Xi \in \mathcal{X}_q \) coincides with the admissible output multiplicative perturbation to be considered for Problem 2, and \( \Gamma, W, R, Q, H, L \) denote just the subcompensators that are obtained by applying Procedure 2 to system \( \Sigma \);
- (ii) therefore, the control system \( \tilde{\Sigma}^d \) depicted in figure 3(b) coincides with the control system \( \tilde{\Sigma} \) depicted in figure 2.

4. Examples

In this section, the proposed design procedures are illustrated through two examples. The first one is concerned with an SISO plant, so that it is easy to appreciate the advantages of using periodic control, whereas the second example is concerned with an MIMO plant.
Example 1: Consider the LTI reachable and observable SISO plant \( S \) characterized by
\[
A = \begin{bmatrix} 7/2 & -1 & -3/2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \\
C = \begin{bmatrix} 3 & -5 & 1/2 \end{bmatrix}, \quad D = [1].
\]

It can be computed that such a plant has the transfer function
\[
E(z) = \frac{(z - 2)(z + 2)(z - 1/2)}{(z + 1/2)(z - 1)(z - 3)}.
\]

Since there are two zeros out of the unit circle, it is evident, by a root-locus argument, that, for any LTI feedback compensator, if the unknown gain \( c_1 \) of the corresponding perturbed plant \( S_{out}^1 \) (or \( S_{out}^2 \)) tends to infinity, the closed-loop system becomes unstable. Therefore, the infinite gain margin problem is not solvable by an LTI compensator.

Since the plant is SISO, for any fixed \( \rho \), each compensator solving Problem 1 also solves Problem 2, and vice versa. In this example, the application of Procedure 2 to obtain a compensator solving both Problems 1 and 2, with \( \rho = 0.8 \), is illustrated.

As in any SISO plant for which Problem 2 can be solved, at Step 2.1 we obtain \( c = 1 \), \( C_0 = \{1\} \) and \( R_0 = \{1\} \). At Step 2.2, we have \( V = C \), \( v = 3 \), hence \( v + c = 4 \). It can be easily verified that the pair \( (A^4, V) \) is reconstructible, hence \( \omega = 4 \). At Steps 2.3–2.5, we obtain \( L_0 = 1 \), \( R_0 = -1 \), \( Q_0 = 0 \), and \( L_i = R_i = Q_i = 0 \), \( i = 1, \ldots, 3 \). At Step 2.6, \( W_0 = 0 \), and the only matrix \( K \) such that the eigenvalues of \( A^4 + KV \) are all equal to zero is given by
\[
K = \begin{bmatrix} 196814611 \\ 3330000 \\ 32812301 \\ 1665000 \\ 5473651 \\ 832500 \end{bmatrix} \approx \begin{bmatrix} -59.1035 \\ -19.7071 \\ -6.5750 \end{bmatrix}
\]

then
\[
U_2 = R_0(1 + Q_0) = -1
\]

\[
B_1 = [B \ AB A^2 B] = \begin{bmatrix} 1 & 7 & 45 \\ 2 & 7 & 4 \\ 0 & 1 & 2 \end{bmatrix},
\]

\[
B_2 = A^3 B = \begin{bmatrix} 275 \\ 8 \\ 45/4 \\ 7/2 \end{bmatrix}
\]

\[
U_1 = B_1^{-1}(K + B_2) = \begin{bmatrix} 498491 \\ 277500 \\ 479782 \\ 208125 \\ 2559901 \\ 832500 \end{bmatrix} \approx \begin{bmatrix} 1.7964 \\ 2.3053 \\ -3.0750 \end{bmatrix}
\]

Figure 3. (a) The control system \( \tilde{\Sigma}_a \) obtained by applying Procedure 1 to \( S^d \) under the hypotheses of Procedure 2; (b) its dual system \( \tilde{\Sigma}_a^d \).
from which it follows that $W_1 \approx -3.0750$, $W_2 \approx 2.3053$ and $W_3 \approx 1.7964$. At Step 2.7, matrix $T$ can be chosen as

$$T = \begin{bmatrix}
  0 & 17989354671300 & 54173142963000 \\
  180852300 & 363023350718371 & 54173142963000 \\
  23620923481 & 363023350718371 & 0 \\
  52462 & 153691 & 23620923481 \\
  153691 & 18724 & 153691
\end{bmatrix}$$

whereas the Jordan form of $A^w + KV$ is a single Jordan block, so that $r = 3$. It can be easily verified that (32) is satisfied for any $\gamma_1 > \gamma_{\text{lm}}$, where $\gamma_{\text{lm}} = 7637.120003$. Therefore the value $\gamma = 7637.13$ can be chosen. By completing the design through Steps 2.8 and 2.9, the resulting compensator is a periodic system of period 4, whose state space dimension is equal to 1 (as for any SISO plant). Moreover, since the only dynamic subcompensator is $H$, and there are no algebraic loops in the compensator itself, it is evident that the asymptotic stability of the compensator depends merely on the characteristic multipliers of $A^H(k)$; hence, since $A_{12}^H A_{21}^H A_{31}^H A_{01}^H = 0$, the compensator has the characteristic multiplier equal to 0 and is therefore dead-beat stable, and hence $\rho$-stable. Starting with the initial state $x(0) = [-1.25, -0.5, 0]'$, three simulations have been performed letting, respectively, $\xi_1 = 1$, $\xi_1 = 5$ and $\xi_1 = 50$. The results have been reported in figure 4, using a different line type for each simulation. From plot (a), it can be noted that the compensation mechanism that is adopted renders the control input very insensitive to the actual value of $\xi_1$, so that the three lines are wholly superimposed, whereas, from plot (b), it is evident that the magnitude of the output

![Figure 4](image-url)
response over some initial time interval strongly depends on the value of $\xi_1$. In order to emphasize that the three lines in plot (a) are actually not coincident, in plot (c) the natural logarithm of the input magnitude is reported, so that it is evident that different convergence rates are actually achieved depending on the actual value of $\xi_1$. Obviously, such different convergence rates can be appreciated also from plot (d), where, for completeness, the natural logarithm of the output magnitude is reported.

Note that, in both examples reported here, all the computations performed in the design procedure before the computation of matrix $T$ have been performed with infinite precision, but the compensator used in the simulations is described by the approximated matrices (with four decimal digits).

**Example 2:** Consider the LTI reachable and observable MIMO plant $S$ characterized by

$$A = \begin{bmatrix} 16 & 13 & -1 & 0 & 22 & 13 \\ 5 & 15 & 0 & 0 & 0 & 15 \\ 0 & 31 & 4 & 0 & 44 & 26 \\ 15 & 5 & 0 & 5 & 15 \\ -4 & 13 & 2 & 6 & -33 & 13 \\ 15 & 5 & 0 & 0 & -1 & 0 \\ 0 & 26 & 0 & 0 & 0 & 8 \\ 15 & 5 & 0 & 0 & 0 & 8 \\ 15 & 5 & 0 & 0 & 0 & 8 \\ 15 & 5 & 0 & 0 & 0 & 8 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 3 \\ -2 & 1 & 2 \\ 2 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 2 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}.$$  

Since the Jordan form of matrix $A$ has three Jordan blocks relative to the unstable eigenvalue $6/5$, it is easily seen that system $S$ satisfies both Assumptions 1 and 2; moreover, it also satisfies conditions (ii) of Theorems 1 and 2. Now, the application of Procedure 1 to obtain a compensator solving Problem 1, with $\rho = 1$, will be illustrated. Note that Procedure 2 could also be applied, thus obtaining a different compensator solving Problem 2 for the same nominal plant.

At Step 1.1, one possible choice of the integer $c$ is $c = 2$, with $C_1 = \{1, 2\}$, $C_2 = \{3\}$, $R_1 = \{1, 2\}$ and $R_2 = \{2\}$. At Step 1.2, we have

$$V = \begin{bmatrix} -1 & 0 & -2 & 2 & 1 & 0 \\ 1 & 0 & 1 & -1 & 0 & 0 \\ 1 & 41 & 2 & 1 & 0 & 46 \\ \frac{3}{15} & \frac{3}{15} & \frac{-3}{15} & 0 & \frac{-15}{15} \end{bmatrix}$$

and the observability index is $\nu = 2$, hence $\nu + c = 4$. It can be easily verified that the pair $(A^4, V)$ is controllable, hence $\omega = 4$. At Steps 1.3–1.5, we obtain

$$L_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$L_1 = L_0 = 0_{3 \times 3}$$

$$R_3 = \begin{bmatrix} 0 & -1 & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$R_1 = R_0 = 0_{3 \times 3}$$

$$Q_3 = 0_{3 \times 3}, \quad Q_2 = L_3 R_3 CB = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{2} & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Q_1 = Q_0 = 0_{3 \times 3}.$$  

At Step 1.6, the matrix $K$ such that the eigenvalues of $A^4 + KV$ are all equal to zero can be chosen as

$$K = \begin{bmatrix} 0 & 1679616 & 1679616 & 1679616 & 1679616 & 1679616 \\ 419375 & 419375 & 419375 & 625 & 419375 \\ 23976 & 0 & 2592 & 1944 & 1944 & 0 \\ 5125 & 0 & 0 & 26621659 & 3125 & 0 \\ 3125 & 0 & 3125 & 3125 & 0 & 0 \\ 306188141 & 0 & 306188141 & 332184625 \\ 0.0 & -4.0050 & -4.0050 & -3.0736 & -4.0050 \\ 0.0 & 4.1472 & 0.6221 & -0.6221 & 0.0 \\ -7.6723 & 0.0 & 0.0 & 0.0 & -0.8694 \end{bmatrix}$$

then

$$U_2 = (I_3 + Q_2) R_2, \quad C_1 = \begin{bmatrix} C \\ CA \end{bmatrix}, \quad C_2 = \begin{bmatrix} CA^2 \\ CA^3 \end{bmatrix}.$$
and $U_1 = (K - U_2 C_2) C_1^{-1}$, and, finally, letting $U_1 = [W_0 \ W_1]$, we obtain

$$W_0 \approx \begin{bmatrix} 0.5081 & -0.5760 & -1.5404 \\ 0.1200 & -4.3546 & -0.0000 \\ -0.3951 & 1.4400 & -0.0003 \end{bmatrix} \quad \text{and} \quad W_1 \approx \begin{bmatrix} -0.4234 & -0.4800 & 4.6212 \\ 0.6200 & 4.1472 & 0.0000 \\ -0.3951 & -2.4000 & 0.0003 \end{bmatrix}$$

whereas $W_2 = W_3 = 0_{3x3}$. At Step 1.7, matrix $T$ can be chosen as

$$T = \begin{bmatrix} 0 & 1902704375 & 0 & 0 & 0 \\ 377494535 & 10439712864 & 0 & 0 & 0 \\ 3355621992 & 2 & 0 & 0 & 0 \\ 14178125 & 0 & 3125 & 0 & 0 \\ 12083001 & 0 & 1944 & 0 & 0 \\ 11251700 & 0 & 10553287 & 12083001 & 0 \\ -1 & 8924629690028875 & 0 & 3125 & 0 \\ -1 & 1409039069228768 & 0 & 1944 & 0 \\ -1 & 5261220436787 & 0 & 5738777 & 1726143 \\ 2174443050816 & 0 & 5755381 & 575381 & 2174443050816 \end{bmatrix}$$

whereas the Jordan form of $A^w + VK$ is constituted by three Jordan blocks of dimension 2, so that $\tau = 2$. It can be easily verified that (14) is satisfied for $i = 1, 2, 3$. for any $\gamma_1 > \gamma_{1m}$, $\gamma_2 > \gamma_{2m}$ and $\gamma_3 > \gamma_{3m}$, where $\gamma_{1m} = 6053.3369$, $\gamma_{2m} = 8468.8063$ and $\gamma_{3m} = 3539.8231$. Therefore the values $\gamma_1 = 6053.4$, $\gamma_2 = 8468.9$ and $\gamma_3 = 3539.9$ can be chosen. By completing the design through Steps 1.8 and 1.9, the resulting compensator is a periodic system of period 4, whose state space dimension is equal to 3. Moreover, since the only dynamic subcompensator is $\tilde{H}$, there are no algebraic loops in the compensator itself, it is evident that the asymptotic stability of the compensator depends merely on the characteristic multipliers of $A^H(k)$; hence, since $A^H(3)A^H(2)A^H(1)A^H(0) = 0_{3x3}$, the compensator has all the three characteristic multipliers equal to 0 and is therefore dead-beat stable, and hence $\rho$-stable.

Starting with the initial state $x(0) = [-0.2, -0.1, -1.2, 2.2, 1, -0.9]^T$, three simulations have been performed letting, respectively, $[\xi_1, \xi_2, \xi_3] = [1, 1, 1]$, $[\xi_1, \xi_2, \xi_3] = [2, 3, 5]$ and $[\xi_1, \xi_2, \xi_3] = [10, 25, 20]$. The results have been reported in figures 5, 6 and 7 using a different line type for each simulation (in the first two figures a shorter time interval has been considered, in order to emphasize the relevant information). From figure 5, it is evident that the magnitude of the control inputs response over some initial time interval strongly depends on the values of the unknown scalar gains; contrary to what happens to the output of the SISO plant considered in Example 1, here the magnitudes of the control inputs applied during the initial time interval decrease when the unknown scalar gains assume higher values: this is expected, since the compensator structure is aimed to counteract the increase of the scalar gains, so that, when larger gains are present, the compensator generates smaller control signals. On the other hand, from figure 6 it can be verified that the compensation mechanism that is adopted renders the output responses very insensitive to the actual values of $\xi_1$, $\xi_2$ and $\xi_3$, so that the lines representing the time response of the outputs are wholly superimposed. In order to emphasize that the three lines in each of the plots of figure 6 are actually not coincident, in plot (b) of figure 7 the natural logarithm of the Euclidean norm of the output response is reported, so that it is evident that different convergence rates are actually achieved depending on the actual value of the unknown gains. Obviously, such different convergence rates can be appreciated also from plot (a) of the same figure, where, for completeness, the natural logarithm of the norm of the input response is reported. Note that in plot (a) of figure 7, some values of the logarithm of the norm of the input are not reported, since at those times the applied input is exactly zero, since $L_0 = L_1 = 0_{3x3}$.

5. Conclusions

A complete solution of the problem of the infinite gain margin strong stabilization with prescribed rate of convergence of the free responses of the closed-loop control system has been given for MIMO linear time invariant systems controlled by periodic linear controllers.
Figure 5. Time response of the input variables for the three simulations described in Example 2. The continuous lines are relative to \([\xi_1, \xi_2, \xi_3] = [1, 1, 1]\), the dashed lines to \([\xi_1, \xi_2, \xi_3] = [2, 3, 5]\), the dotted lines to \([\xi_1, \xi_2, \xi_3] = [10, 25, 20]\).

Figure 6. Time response of the output variables for the three simulations described in Example 2. The same notations as in figure 5 are used.
The necessary and sufficient conditions derived in this paper have been compared with the stronger sufficient conditions already in the literature. In particular, the results in this paper confirm a conjecture that was formulated in Khargonekar et al. (1985) about the necessity of non-zero direct feed-through terms in order to ensure infinite gain margin stabilization for SISO plants. Constructive procedures have been given for designing solutions to the problems under consideration, both for unknown scalar gains acting on the inputs of the plant, and for unknown scalar gains acting on the outputs of the plant.

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References


