On the Notion of Transfer Function for Linear Hybrid Systems with Periodic Jumps

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Abstract—In this paper, we introduce the notion of transfer function for linear hybrid systems in the presence of time-driven periodic jumps. Such concept is employed to extend frequency-domain analysis tools — well-known and extensively used in the context of purely continuous or discrete time systems — to a class of hybrid systems. In particular, knowledge of the transfer function permits firstly the computation in closed-form of the complete forced response of the hybrid plant. Moreover, the hybrid transfer function allows for the frequency-domain definition of the notion of 0-th moment of the underlying system, which is instrumental, e.g., for the solution to model reduction or system identification problems. It is shown, in addition, that the notion of moment based on the hybrid transfer function possesses an interesting time-domain counterpart, hence extending the results obtained for purely continuous-time systems. The paper is concluded by numerical simulations to further substantiate the theoretical claims.

I. INTRODUCTION

The study of hybrid systems, characterized by the interaction between a continuous-time evolution (typically referred to as the “flow” dynamics) and a discrete-time behavior (here named “jump” dynamics), has gained increasing attention in the last years, with several important results [1]–[5]. The interest is mainly motivated by the interplay of continuous processes and discrete time events, peculiar feature of hybrid systems, which permits the modeling of most control practical applications characterized by the presence of an analog plant controlled by a digital device, or the behavior of physical systems that experience impulsive events [6]–[9].

In light of such interest, several classical problems in control theory have been recently revisited and adapted to the context of hybrid systems with periodic jumps, including, for instance, stabilization and optimal control [10], [11], output regulation, see [12]–[15] for the nominal case and [16] for an approach to robust internal model-based regulation, dynamic games [17] as well as the characterization of structural properties [18]. All of the above solutions provide time-domain (state-space) characterizations of the control task, while the frequency-domain (transfer function) approach has been overlooked, despite its well-known usefulness in providing deep insights on the behavior of non-hybrid systems.

The main contribution of the paper is twofold. Firstly, we introduce the notion of transfer function for linear hybrid systems in the presence of time-driven periodic jumps. Then, we exploit such concept to extend some of the machinery associated to frequency-domain analysis of purely continuous/discrete-time systems to the hybrid context. This permits, to begin with, the explicit computation, in closed-form, of the complete forced response of the hybrid plant, distinguishing in addition between forced and free as well as transient and steady-state responses. Then, we provide the frequency-domain definition of 0-th moment of the hybrid plant, based on the knowledge of the hybrid transfer function, and we show that, similarly to the non-hybrid, case, such definition possesses a well-defined time-domain counterpart.

II. LINEAR HYBRID SYSTEMS WITH PERIODIC JUMPS

Notation. Let \( \mathbb{R}, \mathbb{R}_0, \mathbb{Z}, \mathbb{Z}_{\geq 0} \) and \( \mathbb{C} \) be the set of real, nonnegative real, integer, natural and complex numbers, respectively. The symbol \( i \) denotes the imaginary unit, \( i = \sqrt{-1} \). Given \( c \in \mathbb{C} \), \( \Re(c) \) and \( \Im(c) \) denote its real and imaginary part, respectively. Let \( \mathcal{C}_0 := \{ s \in \mathbb{C} : |s| < 1 \} \).

Let \( A \in \mathbb{R}^{n \times n} \), the symbol \( \Lambda(A) \) denotes the spectrum of the matrix \( A \). A set \( \mathcal{T} \subset \mathbb{R}_0 \times \mathbb{Z}_{\geq 0} \) is an hybrid time domain if, for each \( (\tau, \kappa) \in \mathcal{T} \), the set \( \mathcal{T} \cap [0, \tau] \times \{0, 1, \ldots, \kappa\} \) equals \( \bigcup_{k=0}^{\kappa} I_k \times \{k\} \), where \( I_k = [t_k, t_{k+1}] \), \( k = 0, \ldots, \kappa \), and \( t_0 \leq t_1 \leq \cdots \leq t_\kappa \leq t_{\kappa+1} = \tau \).

Let \( \tau_M \in \mathbb{R}_0 \) and consider the hybrid time domain

\[
\mathcal{T} = \bigcup_{k=0}^{\infty} [k\tau_M, (k+1)\tau_M) \times \{k\},
\]

and the LTI hybrid system governed by the flow dynamics

\[
\dot{x} = Ax + Bu,
\]

and subject to jumps according to

\[
\dot{x} = Ex + Fv,
\]

with \( x(t, k) \in \mathbb{R}^n \), \( u(t, k) \in \mathbb{R}^m \) and \( v(k) \in \mathbb{R}^\mu \), and where \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, E \in \mathbb{R}^{n \times n}, \) and \( F \in \mathbb{R}^{n \times \mu} \). Despite the fact that, for generic hybrid systems, each solution is defined over its own hybrid time domain, the class of hybrid systems considered in this paper is characterized by the fact that all the hybrid arcs are defined on the same hybrid time domain \( \mathcal{T} \), that is therefore fixed a priori. Such assumption, in fact, preserves linearity of the overall hybrid system and permits the frequency characterization of (1) discussed in the following.

Namely, letting \( t_k = k\tau_M \), given an hybrid arc \( u : \mathcal{T} \rightarrow \mathbb{R}^m \) (i.e., a function such that, for each fixed \( k \in \mathbb{Z}_{\geq 0} \), the map \( t \mapsto u(t, k) \) is locally absolutely continuous on \( [t_k, t_{k+1}] \) and \( v : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}^\mu \), an hybrid arc \( x : \mathcal{T} \rightarrow \mathbb{R}^n \) is a solution to system (1) with inputs \( u \) and \( v \) if

\[
\frac{d}{dt} x(t, k) = Ax(t, k) + Bu(t, k) \quad \text{for almost all } t \in [t_k, t_{k+1}],
\]

and

\[
x(t_{k+1}, k+1) = Ex(t_k, k) + Fv(k),
\]

for all \( k \in \mathbb{Z}_{\geq 0} \).
Following the notation of [18]–[20], let $\varphi(t,k,x_0,u,v)$ be the (unique) solution to system (1) at hybrid time $(t,k) \in \mathcal{T}$, with initial condition $x_0 \in \mathbb{R}^n$ and inputs $u$ and $v$. System (1) is Linear Time Invariant (briefly, LTI), in the sense that, given two initial conditions $x_{0,1}, x_{0,2} \in \mathbb{R}^n$, two hybrid arcs $u_1, u_2$, and two functions $v_1, v_2$, one has that $\varphi(t,k,x_{0,1},\alpha x_{0,1} + \beta x_{0,2},\alpha u_1 + \beta u_2,\alpha v_1 + \beta v_2) = \alpha \varphi(t,k,x_{0,1},u_1,v_1) + \beta \varphi(t,k,x_{0,2},u_2,v_2)$, for each $\alpha, \beta \in \mathbb{R}$ and $(t,k) \in \mathcal{T}$, and that the solution is independent of the initial time. System (1) is asymptotically stable if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $|x_0| < \delta$ implies $|\varphi(t,k,x_0,0)| < \varepsilon$ and $\lim_{t \to \infty} \varphi(t,k,x_0,0) = 0$ for each $x_0 \in \mathbb{R}^n$. The monodromy matrix of system (1) is $\tilde{E} := E e^{A\tau M}$. The eigenvalues of such a matrix are strongly related with the stability properties of system (1). Namely, by [20, Lem. 1], system (1) is asymptotically stable if and only if $\Lambda(\tilde{E}) \subseteq \mathbb{C}_g$. In the following, given an hybrid arc $\xi : \mathcal{T} \to \mathbb{R}^n$, the symbol $\xi_{k\downarrow}$ denotes the post-jump “sampled” values $\xi(t,k)$.

The following proposition provides a formula to compute the (unique) solution $\varphi(t,k,x_0,u,v)$ given $u$, $v$, $x_0$.

**Proposition 1.** Consider the linear hybrid system (1) and suppose that $u(t,k)$, $v(k)$ and $x_0$ are given. Then, for all $(t,k) \in \mathcal{T}$, one has that

$$
\varphi(t,k,x_0,u,v) = \int_{t_k}^{t} e^{A(t-\tau)}Bu(\tau,k)d\tau + e^{A(t-t_k)}(E_{k}\dot{x}_0 + \sum_{j=0}^{k-1}(\dot{E}_{k-1-j}Fv(k)) + \sum_{j=1}^{k-1}\dot{E}_{k-1-j}E\int_{t_j}^{t_{j+1}} e^{A(\tau-\tau)}Bu(\tau,j)d\tau).
$$

It is worth noticing that the formula given in (2) to compute the solution to system (1) with inputs $u$ and $v$ may be hardly employed in practice. As a matter of fact, if, e.g., $u(t,k) \neq u(t + \tau_M, k + 1)$, $k \in \mathbb{Z}_{\geq 0}$, one has to compute integrals of the form $\int_{t_j}^{t_{j+1}} e^{A(t-\tau)}Bu(\tau,j)d\tau$, $j = 1, \ldots, k$, that appear to be computationally intractable in practice for $k$ large enough. To circumvent this issue and to provide interesting insights on the structure of the solutions to (1), in Section IV, we propose an alternative method to determine the solution to system (1) based on a hybrid extension of the classical concept of transfer function formally defined in the following section.

### III. A TRANSFORM FOR HYBRID ARCS

Before providing the formal definition of hybrid transfer function, let us recall few basic definitions and results concerning Laplace and Z transforms. Let $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ be a given locally integrable function. The Laplace transform of $f(t)$ (denoted with capital letters) is

$$
F(s) = \mathcal{L}\{f(t)\}_{t \to s} := \int_{0}^{\infty} f(t)e^{-st}dt,
$$

where $s \in \mathbb{C}$, provided that the integral on the right hand side exists. The subset of $\mathbb{C}$ where the integral above is well-defined is called domain of convergence. Given a Laplace transform $F(s)$, the inverse Laplace transform [21] is

$$
f(t) = \mathcal{L}^{-1}\{f(s)\}_{s \to t} := \frac{1}{2\pi i} \lim_{\gamma \to +\infty} \int_{\gamma - iT}^{\gamma + iT} F(s)e^{st}ds,
$$

where $\gamma \in \mathbb{R}$ is in the domain of convergence of $F(s)$. On the other hand, the Z transform of $g : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}$ is

$$
G(z) = \mathcal{Z}\{g(k)\}_{k \to z} := \sum_{k=0}^{\infty} g(k)z^{-k},
$$

where $z \in \mathbb{C}$. The subset of the complex plane $\mathbb{C}$ where the sum on the right hand side of the expression above converges is called domain of convergence. Given a Z transform $G(z)$, the inverse Z transform is given by

$$
g(k) = \mathcal{Z}^{-1}\{G(z)\}_{z \to k} := \frac{1}{2\pi i} \oint_{S} G(z)z^{k-1}dz,
$$

where $S$ is any counterclockwise closed path contained in the domain of convergence of $G(z)$. Let an hybrid arc $\xi : \mathcal{T} \to \mathbb{R}$ be given. The hybrid (briefly, H) transform of $\xi(t,k)$ is

$$
\mathcal{H}\{\xi(t,k)\}_{t \to s,k \to z} = \mathcal{Z}\{\xi(\sigma + k\tau_M,k)\}_{k \to z_{\sigma \to s}},
$$

where $s$ and $z$ belong $\mathbb{C}$. The subset of $\mathbb{C} \times \mathbb{C}$ where the sum and the integral involved in the definition given in (3) are well defined is called Domain Of Convergence (briefly, DOC). Since the H transform is essentially the combination of Z and Laplace transforms, many of the properties of these operators are preserved by the H transform as, e.g., linearity. Given an H transform $\Xi(s,z)$, the inverse H-transform is

$$
\xi(t,k) = \mathcal{H}^{-1}\{\Xi(s,z)\}_{s \to t,k \to k} = \mathcal{Z}^{-1}\{\mathcal{L}^{-1}\{\Xi(s,z)\}_{s \to t-k\tau_M}\}_{z \to k}.
$$

The following two lemmas provide results on the H-transform of an hybrid arc defined above.

**Lemma 1.** Let $\xi : \mathcal{T} \to \mathbb{R}$ and assume that there exists $c, \lambda, g \in \mathbb{R}_{\geq 0}$ such that $|\xi(t,k)| \leq ce^{Mg^k}, \forall (t,k) \in \mathcal{T}$. Then the set $\{(s,z) \in \mathbb{C} \times \mathbb{C} : \mathcal{R}(s) > \lambda, |z| > ge^{\lambda\tau_M}\}$ is a subset of the domain of convergence of $\Xi(s,z)$.

**Lemma 2.** Assume that there exists $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^h$ and $g : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}^h$ such that $\xi \sigma + k\tau_M, k \rightarrow \mathcal{Z}\{\xi(\sigma + k\tau_M,k)\}_{k \to z_{\sigma \to s}} = \mathcal{Z}\{\xi(\sigma + k\tau_M,k)\}_{\sigma \to s_{k \to z}} = \mathcal{L}\{f(\sigma)\}_{\sigma \to s}\mathcal{Z}\{g(k)\}_{k \to z}$, and, letting $C$ and $D$ be the region of convergence of $\mathcal{L}\{f(\sigma)\}_{\sigma \to s}$ and of $\mathcal{Z}\{g(k)\}_{k \to z}$, respectively, the region of convergence of $\mathcal{H}\{\xi(t,k)\}_{t \to s,k \to z}$ is $C \times D$.

To further substantiate the definitions introduced above, consider the following numerical examples of H-transform of two hybrid arcs that find useful applications in engineering.
Example 1. (i) Consider the hybrid arc \( \xi(t,k) = e^{\lambda t} g^k \). Thus, \( \xi(\sigma + k\tau_M,k) = e^{\lambda \sigma} (g e^{\lambda \tau_M})^k \) and hence, by Lemma 2
\[
H(\xi(t,k)) \quad \text{for} \quad t \rightarrow s, k \rightarrow z
\]
and the DOC is \( \{ (s,z) \in \mathbb{C}^2 : \Re(s) > \lambda, |z| > \Re(e^{\lambda \tau_M}) \} \).

(ii) Consider the hybrid arc \( \xi(t,k) = \sin(\omega_1 t + \omega_2 k) \). One has that \( \xi(\sigma + k\tau_M,k) = \sin(\omega_1 \sigma) + (\omega_1 \tau_M + \omega_2 k) = \left[ \sin(\omega_1 \sigma) \cos(\omega_1 | \omega_2 | k) \right] + \left[ \cos(\omega_1 \tau_M + \omega_2 k) \cos(\omega_1 \tau_M + \omega_2 k) \right]^T \).

\[
H(\xi(t,k)) \quad \text{for} \quad t \rightarrow s, k \rightarrow z
\]
and the DOC is \( \{ (s,z) \in \mathbb{C}^2 : \Re(s) > 0, |z| > 1 \} \).

IV. HYBRID TRANSFER FUNCTION

In order to introduce the input–output transfer function of the hybrid system (1), define the output
\[
y(t,k) = C x(t,k), \quad (4)
\]
where \( C \in \mathbb{R}^{p \times n} \). Clearly, the hybrid arc \( y : T \rightarrow \mathbb{R}^p \) can be obtained, in principle, by (2) as \( y(t,k) = C \varphi(t,k,x_0, u, v) \) for all \( (t,k) \in T \). However, as highlighted at the end of Section II, such an approach may be intractable in practice. Therefore, in this section, we introduce the concept of hybrid transfer function by employing the H-transform introduced in Section III.

Definition 1. Consider system (1) with output (4). Then, the hybrid transfer function of system (1), (4) is the tuple
\[
W(\ell, s, z) = \{ W_0(\ell,z), W_a(\ell), W_b(\ell,s,z), W_c(\ell,z) \}, \quad (5)
\]
where, letting \( I \) be the identity matrix and \( \bar{E} = E e^A \tau_M \),
\[
W_0(\ell,z) = C((I - A)^{-1} z(I - \bar{E})^{-1},
W_a(\ell) = C((I - A)^{-1} B,
W_b(\ell,s,z) = C((I - A)^{-1} z(I - \bar{E})^{-1} E(sI - A)^{-1} B,
W_c(\ell,z) = C((I - A)^{-1} z(I - \bar{E})^{-1} F.
\]

The hybrid transfer function defined in (5) allows to easily compute the output \( y(t,k) \) of system (1) with inputs \( u \) and \( v \) as stated in the following theorem.

Theorem 1. Let \( u : T \rightarrow \mathbb{R}^m, v : Z \geq 0 \rightarrow \mathbb{R}^n \) and \( x_0 \in \mathbb{R}^n \) be given. Let \( U(\ell,z) = H\{u(t,k)\}_{t \rightarrow s, k \rightarrow z} \) and \( V(z) = \{ v(k) \}_{k \rightarrow z} \). Let
\[
Y(\ell,z) = W_0(\ell,z) x_0 + W_a(\ell) U(\ell,z)
\]
\[
+ \mathcal{L}^{-1} \{ W_b(\ell,s,z) U(s,z) \}_{s \rightarrow \tau_M} + W_c(\ell,z) V(z).
\]
Then, \( y(t,k) = H\{-Y(\ell,z)\}_{t \rightarrow s, z \rightarrow k} \).

Remark 1. It is worth noticing that there is a one–to–one correspondence among the terms in (2) and the four entries of the transfer function \( W(\ell,s,z) \) given in (5). Namely, \( W_a(\ell) \), \( W_0(\ell,z) \), \( W_c(\ell,z) \), and \( W_b(\ell,s,z) \) corresponds to the first, second, third and fourth term of the expression given in (2), respectively. Furthermore, the dependency on \( s \) of \( W_b(\ell,s,z) \) has been introduced in order to take into account the convolutions that appear in the last term of the expression given in (2) that essentially relates the input \( u : T \rightarrow \mathbb{R}^m \) with the monodromy behavior of system (1).

V. STEADY-STATE AND HARMONIC RESPONSES OF HYBRID SYSTEMS

The aim of this section consists in comprehensively characterizing the steady-state, state or output, response of a linear hybrid system with periodic (time-driven) jumps forced by basic, possibly persistent, inputs, similarly to the standard use of the transfer function in purely continuous or discrete time. Towards this end, we first introduce the input-state transfer function, \( H(\ell,s,z) \), which is essentially defined as the tuple \( W(\ell,s,z) \) by letting \( C = I \). Given an hybrid arc \( \xi : T \rightarrow \mathbb{R} \), in the following, the shortcut \( \xi(\sigma,k) \) is used to denote \( \xi(\sigma + k\tau_M,k) \), when confusion cannot arise.

Theorem 2. Consider linear hybrid systems with periodic jumps described by (1) forced by inputs of the form
\[
u(\sigma,k) = e^{\lambda \sigma} g^k,
\]
with \( \lambda \in \mathbb{C} \) and \( g \in \mathbb{C} \). Suppose that \( \Lambda(E e^A \tau_M) \subset \mathbb{C}_g \). Then, the steady-state output response is defined by
\[
y_{ss}(\sigma,k) = \mathcal{L}^{-1} \left\{ W_a(\ell) \frac{1}{e^{\lambda \ell}} \right\} e^{\lambda \sigma} + C e^{A \sigma} (A_d e^{\sigma \phi_d}) g^k,
\]
with
\[
A_d = \| \bar{H}_b(\lambda, g) \|, \quad (9a)
\]
\[
\phi_d = \left[ \bar{H}_b(\lambda, g) \right], \quad (9b)
\]
where \( \bar{H}_b \) is defined as
\[
\bar{H}_b(\lambda, z) = \lim_{\ell \rightarrow +\infty} \mathcal{L}^{-1} \left\{ H_b(\ell,s,z) \frac{1}{e^{\lambda \ell}} \right\}_{s \rightarrow \tau_M} \quad (10)
\]
Remark 2. As in the classical purely continuous or discrete time, the basic input (7) can be employed to generate sinusoidal inputs at desired frequencies and phases, by allowing for complex values of the pole \( \lambda \) and by exploiting the principle of superposition. This can be achieved, separately or simultaneously, for the continuous-time and discrete-time component of the control law. In fact, it is well-known that \( 2 \cos(\omega \delta) = (e^{i\omega \delta} + e^{-i\omega \delta}) \) and, similarly, \( 2 \sin(\omega \delta) = i(e^{i\omega \delta} - e^{-i\omega \delta}) \), where \( \delta \) may describe equivalently the continuous or discrete time.

Remark 3. Due to the considered nature of the interplay between continuous-time and discrete-time evolutions, i.e. in the presence of time-driven periodic jumps, it is interesting to point out that the steady-state behavior during flows does not coincide, in general, with the steady-state response of the flow dynamics alone (considered as a purely continuous–time system) as it can be immediately appreciated by inspecting the structure of (8). In particular, it is apparent from (8) that the steady-state response during flows contains the natural modes of the flow dynamics, which are not part of the classic steady-state response for continuous-time systems. Note that since the flow dynamics is followed only for a time interval of length \( \tau_M \) after each jump, and hybrid stability (as well as existence of the steady-state response) only depends on the eigenvalues of the monodromy matrix \( \bar{E} \), no assumption is stated on the eigenvalues of the flow dynamics, namely
the poles of the transfer function $W_a(\ell)$, which could be unstable without compromising the validity of the steady-state description in (8) [12], [14], [19], [20].

Remark 4. The presence of additional impulsive inputs $v(k) = \rho^k$ can be easily taken into account and incorporated into the steady-state response (8) by relying on classical results about discrete-time systems. In fact, such steady-state output response should be modified by the additive term given by $Ce^{A\sigma}|W_c(\rho)|e^{\sigma\rho^k}$, with $\rho^k = \sum_{l=0}^{k-1} W_c(\rho)$ and where

$$\hat{W}_c(z) = \lim_{\ell \to \infty} \ell \cdot W_c(\ell, z)$$

is obtained by arguments similar to those around $\hat{H}_b$. ▲

A. Definition of moment of the hybrid transfer function

The above discussions motivate the following definition of the moment of system (1) at a given pair of complex numbers for the continuous-time and discrete-time components of the system. Towards this end, note that the idea consists in associating finite-dimensional information to the underlying hybrid system that allows to reconstruct the monodromy steady-state behavior of the system. This is achieved essentially by considering the steady-state response (8) evaluated for $\sigma = 0$ and for any $k \geq 0$.

Definition 2. The $(\ell, s)$-moment of system (1) at $(s^*, z^*)$ is the complex number $\hat{W}_b(s^*, z^*)$, where

$$\hat{W}_b(\lambda, z) = \lim_{\ell \to \infty} \ell \cdot \mathcal{L}^{-1} \left\{ W_b(\ell, s, z) \frac{1}{s - \lambda} \right\}_{s \to s^*}$$

The notion of moments associated to a given linear system has been extensively exploited in the context of Model Reduction for continuous-time systems. In fact, model reduction by moment matching consists essentially in constructing a lower-dimensional system belonging to the same class of the original system such that the moments of the two systems at a collection of desired complex values (e.g., at desired frequencies in the case of sinusoidal inputs) are identical.

With the objective of extending the same machinery to the hybrid context, or the case in which a LTI system is forced by a discontinuous (piece-wise continuous) input, a definition of moment alternative to the one provided in Definition 2 has been given in [22] and considered also in [23]. The former statement in fact - mimicking the main ideas discussed in the case of nonlinear systems in [24] - defines as moment the entire steady-state response (8) of system (1) forced by a hybrid exosystem of the same form of (1). In particular, briefly recalling the results of [23] (similarly for [22] in which the hybrid arc generated by the exosystem is provided directly in explicit form), let the exosystem $\mathcal{E}$ be defined by the hybrid system with periodic jumps

$$\dot{w} = Sw, \quad (12a)$$
$$w^* = Jw, \quad (12b)$$
$$u = Lw, \quad (12c)$$

with $w(t, k) \in \mathbb{R}^q$. Then, [23, Thm. 1] shows that the steady-state response of the cascade (1)-(12), hence the moment according to [22], [23], is given by

$$y_{ss}(t, k) = C\Pi(\sigma)w(t, k), \quad (13)$$

where the periodic function $\Pi : [0, \tau_M] \to \mathbb{R}^{n \times n}$ is the unique solution of the following system of first-order Ordinary Differential Equation (briefly, ODE) with two-point boundary conditions

$$\Pi(\tau) + \Pi(\tau)S = A\Pi(\tau) + BL, \quad (14a)$$
$$\Pi(0)J = E\Pi(\tau_M). \quad (14b)$$

The following, somewhat negative, result is instrumental for a comparison between the definition $\hat{W}_b(s^*, z^*)$ in the frequency domain based on the notion of transfer function and that based on the steady-state response (8), i.e., (13).

Proposition 2. Consider linear hybrid systems with periodic jumps described by (1), (4), with state of dimension $n$, and suppose that the pairs $(A, B)$ and $(C, A)$ are reachable and observable, respectively. Then, the steady-state response (8) to the input (7) cannot be generated by any LTI hybrid system (1) with state dimension strictly less than $n$. ◁

Advantages and drawbacks of either approaches are discussed in the comparison below. More specifically, Proposition 2 entails that model reduction cannot be performed by moment matching according to the definition of moment considered in [22], [23] while obtaining a reduced-order model belonging to the same class of hybrid systems. This claim is additionally testified by the design of lower-dimensional equivalent systems proposed in [23] and [22], in which somewhat dual approaches are pursued. In the former, the reduced-order model is given in terms of a linear hybrid system with time-varying, i.e. infinite dimensional, input matrix, whereas the latter paper considers hybrid systems with time-varying output matrix. On the other hand, clearly the approach on model reduction based on the notion of moment provided in Definition 2 is such that the entire steady-state response is not reproduced by the lower-dimensional system, which in fact matches only the monodromy behavior of the underlying hybrid plant. The above could be an acceptable approximation provided that the constant $\tau_M$ is sufficiently small, or only the response for short times after each jump is of interest in the application at hand.

Finally, it is worth mentioning the particular scenario in which the transfer function $W_a(\ell)$ possesses stable and sufficiently fast modes (at least relatively to the value of $\tau_M$). In this case, the convolution in the first term of the steady-state response (8) can be approximated by assuming that the continuous-time trajectories (almost) reaches its steady-state description within each flowing interval $[0, \tau_M]$, namely

$$L^{-1} \left\{ W_a(\ell) \frac{1}{s - \lambda} \right\}_{s \to \sigma} \approx |W_a(\lambda)|e^{\sigma L_k(\lambda)} \phi^k, \quad (15)$$

with respect to the input (7). In such a circumstance, the information required to completely reproduce the entire steady-state response (8) can be summarized in terms of two complex numbers associated to $W_a$ and $\hat{H}_b$, respectively. ▲
B. Time-domain characterization of moments

Similarly to what is pursued in [24], in this section we provide an equivalent time-domain characterization of the notion of 0-th moment of a hybrid transfer function for almost any \((s^*, z^*) \in \mathbb{C} \times \mathbb{C}\). Towards this end, we introduce a class of canonical exosystems that allow to relate the notions of moment in frequency and time domains. Such exosystems are described by the equations (12) with matrices

\[
S = \begin{bmatrix}
0 & \omega_1 \\
-\omega_1 & 0
\end{bmatrix}, \quad L = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]

\[
J = \begin{bmatrix}
\cos(\tau_M \omega_1 - \omega_2) & -\sin(\tau_M \omega_1 - \omega_2) \\
\sin(\tau_M \omega_1 - \omega_2) & \cos(\tau_M \omega_1 - \omega_2)
\end{bmatrix},
\]

with initial condition \(w(0,0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T\) and generate sinusoidal functions described by \(s^* = \pm \omega_1\) and \(z^* = e^{\pm \omega_2}\) as continuous-time and monodromy modes, respectively.

**Remark 5.** The structure of (12) with the matrices defined in (16) is reminiscent of the exosystem defined in [24, Lemma 1], namely \(\dot{w} = Sw\) with \(S = s^*\), adapted to the hybrid context. It is in fact required, firstly, to define an exosystem at least of dimension two, in order to take the continuous and discrete time frequencies into account, and then to consider directly an exosystem characterized by real matrices, rather than complex, since such matrices are involved in the definition of a first-order ODE (namely, (14)), in place of an algebraic Sylvester equation. Finally, note that the output \(u(\sigma, k)\) of the exosystem (16) has the form \(u(\sigma, k) = \cos(\omega_1 \sigma) \cos(\omega_2 k) - \sin(\omega_1 \sigma) \sin(\omega_2 k)\) and that a system of the form of (16) cannot generate a signal of the form \(\cos(\omega_1 \sigma) \cos(\omega_2 k)\) for \(\omega_1 \neq 0\) and \(\omega_2 \neq 0\). ▲

**Theorem 3.** Consider linear hybrid system (1). Suppose that \(z^* = e^{\pm \omega_2} \notin \mathbb{L}(E^{A\tau_M})\). Then,

\[
\begin{align*}
\Pi_1 &= \frac{1}{4}(\bar{W}_b(\omega_1, e^{i\omega_2}) - \bar{W}_b(-\omega_1, e^{-i\omega_2})), \\
\Pi_2 &= \frac{1}{4}(\bar{W}_b(\omega_1, e^{i\omega_2}) + \bar{W}_b(-\omega_1, e^{-i\omega_2})),
\end{align*}
\] (17a)

where \(\Pi_i\) denotes the \(i\)-th column of the unique solution of the Sylvester equation

\[
\bar{J}e^{S\tau_M} = E^{A\tau_M} \bar{J} + \int_0^{\tau_M} e^{A(\tau_M-\eta)}BLe^{S\eta}d\eta.
\] (18)

VI. NUMERICAL EXAMPLE

Consider a linear hybrid system described by the equations (1) with matrices given by

\[
A = \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
1
\end{bmatrix}, \quad C = \begin{bmatrix}
1 & 0
\end{bmatrix},
\]

\[
E = \frac{1}{4} \begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix}, \quad F = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}, \quad \tau_M = 1.
\] (19)

The transfer function of such a system is given by the tuple defined in (5) with

\[
\begin{align*}
W_0(\ell, z) &= \begin{bmatrix}
\frac{4\ell - 2\ell^2}{(4z - 3)^2} & \frac{4\ell - 2\ell^2}{(4z - 3)^2}
\end{bmatrix}, \\
W_a(\ell) &= \frac{1}{\pi}, \\
W_b(\ell, s, z) &= \frac{(s+1)(\ell+1)}{s^2(4z-3)^2}, \\
W_c(\ell, z) &= 0.
\end{align*}
\] (20a)

Note that \(W_a(\ell)\) is essentially the transfer function of the continuous–time system (1a), whereas the other elements of \(W(\ell, s, z)\) cannot be easily related with the transfer functions of either (1a) or (1b). Consider the control input \(u(\sigma, k) = e^{\lambda \sigma} g^k\) given in (7). The transfer function introduced in (20) can be employed to compute explicitly the complete (forced) response of system (19). Namely, by (6), it can be shown that

\[
y(\sigma, k) = \lambda^2 g^k (-\lambda \sigma + e^{\lambda \sigma} - 1) + \lambda^2(-2\lambda + e^{\lambda(1\sigma+1)} - 1)(\sigma+1)(\ell\ell^\frac{1}{2} - g^k).
\]

On the other hand, by employing (10), one obtains that \(\bar{H}_b(\lambda, z) = \begin{bmatrix}
-2\lambda + e^{\lambda(1\sigma+1)} - 1 \\
\frac{2\lambda e^{\lambda(1\sigma+1)} - 1}{(4z-3)^2}
\end{bmatrix}\) and hence, by (8), the steady–state output response is

\[
y_{ss}(\sigma, k) = -\frac{\lambda \sigma - e^{\lambda \sigma} + 1}{\lambda^2} g^k + C e^{\lambda \sigma} (A_d e^{\phi_d}) g^k,
\]

where \(A_d\) and \(\phi_d\) are defined as in (9). As highlighted in Remark 2, the response computed above can be employed to determine the solution to an input of the form \(u(\sigma, k) = cos(\omega_1 \sigma) \cos(\omega_2 k)\) is

\[
y_{ss}(\sigma, k) = \frac{1 - \cos(\omega_1 \sigma)}{\omega_1^2} \cos(\omega_2 k) + C e^{\lambda \sigma} (\frac{1}{4} \bar{H}_b(\omega_1, e^{i\omega_2}) e^{i\omega_2 k} + \frac{1}{4} \bar{H}_b(-\omega_1, e^{-i\omega_2}) e^{-i\omega_2 k} + \frac{1}{4} \bar{H}_b(\omega_1, e^{-i\omega_2}) e^{i\omega_2 k} + \frac{1}{4} \bar{H}_b(-\omega_1, e^{i\omega_2}) e^{-i\omega_2 k}).
\]

Figure 1 depicts a numerical simulation of the output response \(y(\sigma, k)\) to the input \(u(\sigma, k) = \cos(\omega_1 \sigma) \cos(\omega_2 k)\), the steady–state output response \(y_{ss}(\sigma, k)\) computed above, and the monodromy steady–state output response \(y_{ss,[k]}\) = \(y_{ss}(0, k)\), for \(\omega_1 = 1\) and \(\omega_2 = 3\), that can be computed directly through the 0–moment of system (1) at \((\omega_1, e^{i\omega_2})\).

![Fig. 1. Forced and steady-state response.](image-url)
of $\omega_1$ and $\omega_2$ provides information about system (1)-(19) essentially identical to the insight given by the Bode plot for classical continuous- and discrete-time systems. Figure 2 depicts $|\tilde{W}_b(\omega_1, e^{j\omega_2})|_{dB} = 20 \log_{10}(|\tilde{W}_b(\omega_1, e^{j\omega_2})|)$ and $\tilde{W}_b(\omega_1, e^{j\omega_2})$ for $\omega_1 \in [0, 20]\frac{\text{rad}}{s}$ and $\omega_2 \in [-\pi, \pi]\frac{\text{rad}}{s}$.

![Moment plot](image)

Fig. 2. Moment plot.

The expressions given in (17) have been used to compute $C\tilde{\Pi}_1$ and $C\tilde{\Pi}_2$ for $\omega_1 = 1$ and $\omega_2 = 3$, by evaluating the transfer function of system (1)-(19) at $\omega_1$ and $\omega_2$, obtaining

$$
C\tilde{\Pi}_1 = 0.103309
$$

$$
C\tilde{\Pi}_2 = -0.178572
$$

which, as expected, match the corresponding values derived by computing $\Pi(\sigma)$ as in (18), namely

$$
\Pi(0) = \begin{bmatrix}
0.103309 & -0.178572 \\
0.103309 & -0.178572
\end{bmatrix}
$$

thus showing the relation between the hybrid transfer function and the 0–moment of the system.

**VII. CONCLUSIONS**

In this paper, the concept of transfer function for linear hybrid systems with periodic jumps has been introduced. It has been shown that, by exploiting the concept of hybrid transform of an hybrid arc, the knowledge of such a function allows to easily compute the solution to the free and forced response of the system in closed form. Furthermore, the link between the hybrid transfer function of a system and its 0–th moment (that has been proven useful for the solution to model reduction and system identification problems) has been highlighted. It is worth noticing that the 0–th moment can be hardly characterized by using a state–space description of the hybrid system. A numerical example has been given to validate the theoretical contributions of the paper.

**REFERENCES**