Transient Optimization in Output Regulation via Feedforward Selection and Regulator State Initialization

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Abstract Output regulation is typically enforced by the use of Internal Model based regulators, which guarantee that steady-state performance (asymptotic tracking and disturbance rejection) is achieved. However, although the overall performance of the regulated system is essentially linked to the properties of the transient responses of such regulators, only a limited attention has been devoted in the literature to such properties, especially in the MIMO case. In this paper, we address the problem of optimizing the transient responses of an already designed closed-loop system including an internal model based output regulation device, by the use of two mechanisms: the selection of an optimized feedforward gain, and the initialization of the states of the regulator. These two mechanisms can be employed both combined or separately, depending on the available information, the control objectives and other application-related constraints.

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1. INTRODUCTION

The classic problem of output regulation has been largely studied for linear Davison (1976); Francis and Wonham (1976) and nonlinear Byrnes et al. (1997); Huang (2004); Serrani et al. (2001) systems. Recently results of output regulation for linear hybrid systems highlighted intriguing new phenomena and structural properties Marconi and Teel (2013); Carnevale et al. (2012a,b, 2013, 2016). In the classical control framework it is well known that, if properly designed, the use of a feedforward controller might sensibly improve the performance of the control system. Feedforward action can improve transients, steady-state performance, and disturbance attenuation in case of measured disturbances. It has been implemented since long time in many applications Seborg et al. (1989); Nisenfeld and Miyasak (1973) where it was a necessary element to solve practical control problems. Given that the classical structure of feedforward design do not change closed-loop stability properties, the design of the feedforward term is usually performed independently by the closed-loop controller Guzmán et al. (2011) and, in common applications, it is a constant gain matrix possibly endowed with lead-lag filters. In Hast and Heggland (2012) strategies for feedforward design having the latter structure has been proposed in order to minimize the $L_2$-norm of the error, mainly referring to the open-loop scheme. Among the relevant contributions to the study of the combination of feedforward and feedback controllers, it is also worth mentioning the work by L. Y. Pao, see e.g. Zhong et al. (2012), which has been tested on several applications, and the work by E. Zattoni and G. Marro, which has mainly been focused on $H_2$ properties, see e.g. Marro and Zattoni (2005).

In this paper we provide a constant matrix feedforward design and controller initialization for linear continuous time systems within the framework of the output regulation and, in order to improve the overall closed-loop performance, the design is performed to minimize specific cost functionals. Unlike open-loop approaches to design feedforward compensators, we propose a feedforward gain matrix $K_f$ depending on the plant $\Sigma_P$, the closed-loop stabilizing controller $\Sigma_C$, the internal model $\Sigma_{IM}$ necessary to ensure (robust) asymptotic tracking of the signal $w$ generated by the exosystem $\Sigma_S$, and their initial state (see Fig. 1). Consequently, the optimal $K_f$ is a function of the overall plant data and its initial state (we provide also results without the knowledge of the initial states) that can even be exploited to properly set the initial state of the overall regulator $(\Sigma_C, \Sigma_{IM})$. It is out of the scope of this paper, however such design procedure can be easily adapted in case of output regulation of linear hybrid systems.
systems with known jump times as in Carnevale et al. (2013), where the state after each jump can be considered as the initial states discussed in this framework.

The paper is organized as follows: in Section 2 the problem formulation is introduced and a specific performance index is formulated in Section 3, whereas a useful parametrization is provided in Section 4 followed by the mean results of Section 5, numerical simulations in Section 6 and then the conclusions.

2. PROBLEM FORMULATION

Consider the control scheme depicted in Fig. 1 and described by linear time invariant equations of the form

\[ \Sigma_P : \begin{align*} 
\dot{x}_p &= A_p x_p + B_p u, \\
y_p &= C_p x_p, 
\end{align*} \]

\[ \Sigma_C : \begin{align*} 
\dot{x}_c &= A_c x_c + B_c e, \\
y_p &= C_c x_p + D_c e, 
\end{align*} \]

\[ \Sigma_{IM} : \begin{align*} 
\dot{x}_{IM} &= A_{IM} x_{IM} + B_{IM} y_{cIM}, \\
y_{cIM} &= C_{cIM} x_{IM}, 
\end{align*} \]

\[ \Sigma_S : \begin{align*} 
\dot{w} &= S w, 
\end{align*} \]

with state \( x_p(t) \in \mathbb{R}^n \), \( x_c(t) \in \mathbb{R}^n \), \( x_{IM}(t) \in \mathbb{R}^{n_{IM}} \), \( w(t) \in \mathbb{R}^q \) and output error \( e = y_p + Q w = C_p x_p + Q w \), with \( e \in \mathbb{R}^p \). The control input \( u(t) \) belongs to \( \mathbb{R}^m, m \geq p \), and is defined as \( u = K_f w + C_{IM} x_{IM} + C_c x_c. \) \( \Sigma_S \) is the exosystem generating the exogenous references to be tracked (or disturbances to be rejected slightly modifying (1)), whereas \( K_f \in \mathbb{R}^{m \times q} \) is the constant matrix that defines the feedforward term \( u_{ff} \). By considering the aggregate state \( x = [x'_p, x'_c, x'_{IM}] \), the interconnection (1a)-(1d) can be rewritten in compact form as

\[ \dot{x} = \begin{bmatrix} A_a & A_{ab} & A_{ba} \\ A_{ba} & A_b \\ A_{ab} & A_b \\ \end{bmatrix} x + \begin{bmatrix} P_a \\ P_b \\ \end{bmatrix} w = Ax + Pw, \]

\[ \begin{bmatrix} A_a & A_{ab} \\ A_{ba} & A_b \\ \end{bmatrix} \begin{bmatrix} A_p + D_{cp} C_p \\ B_p C_{cp} \\ B_p C_{IM} \\ \end{bmatrix} \]

\[ \begin{bmatrix} B_{IM} C_{cIM} \\ B_{IM} \\ \end{bmatrix} + \begin{bmatrix} 0 \\ A_{IM} \\ \end{bmatrix}, \]

\[ \begin{bmatrix} P_a \\ P_b \\ \end{bmatrix} \] \[ \begin{bmatrix} B_p K_f + B_p D_{cp} Q \\ B_p Q \\ B_{IM} D_{cIM} Q \end{bmatrix}, \]

\[ e = [C_p, 0, 0] x + Q w. \]

In the following, we suppose that the controller \( \Sigma_C \) and the internal model unit \( \Sigma_{IM} \) have been suitably designed in order to achieve the output regulation objectives. These include the asymptotic stability property of the origin for the interconnection of the plant \( \Sigma_P \) with \( \Sigma_C \) and \( \Sigma_{IM} \), in the absence of exogenous signals (i.e. \( w = 0 \)), as well as with the ability of the internal model unit to generate the correct steady-state signals\(^1\) to enforce asymptotic convergence of the output error to zero. These properties are recalled in the three following standing assumptions.

**Assumption 1.** The interconnected system (1a)-(1c) with \( w = 0 \) is asymptotically stable, namely \( \Lambda(A) \subset \mathbb{C}^- \).

**Assumption 2.** The condition

\[ \text{rank} \begin{bmatrix} A_a - sI & B_p \\ C_a & 0 \end{bmatrix} = n_p + n_c + p, \]

\[ B_a = [B'_p, 0]' \] and \( C_a = [C_p, 0] \), holds for all \( s \in \Lambda(S) \). As a consequence, for any \( P_a \) and \( Q \) there exist \( \Pi_a = [\Pi'_p, \Pi'_c]' \in \mathbb{R}^{n_p \times n_c} \times q \) and \( \Gamma_a \in \mathbb{R}^{m \times q} \) such that the Francis Equations are satisfied, i.e.

\[ \Pi_a S = A_a \Pi_a + B_a \Gamma_a + P_a, \]

\[ 0 = C_a \Pi_a + Q. \]

\[ \ast \]

Even at this preliminary stage, it is worth stressing the fact that the matrices \( \Pi_a \) and \( \Gamma_a \) solving (4) depends linearly on \( K_f \), which appears in \( P_a \). This parameterization is the topic of a detailed discussion in the following section. Finally, the ability of the internal model unit to generate the correct steady-state signal is translated into the requirements of the following assumption.

**Assumption 3.** The pair \( (A_{IM}, C_{IM}) \) is such that for any \( \Gamma_a \) that solves (4) there exists \( \Pi_{IM} \in \mathbb{R}^{n_{IM} \times q} \) such that

\[ \Gamma_a = C_{IM} \Pi_{IM}, \]

\[ \Pi_{IM} S = A_{IM} \Pi_{IM}. \]

\[ \ast \]

**Remark 1.** The requirement in equations (5) may appear rather strong, but in fact such conditions can be satisfied for any \( \Gamma_a \) provided that the matrix \( A_{IM} \) is selected in such a way that it contains all the modes of the matrix \( S \) in (1d). In particular, equations (4)-(5) consist in the well-known Internal Model Principle for output regulation. \( \blacklozenge \)

The Transient Optimization problem in output regulation can be now formally stated.

**Problem 1.** Find, if any, a matrix gain \( K_f \) and initial conditions \( x_c(0) \) and \( x_{IM}(0) \) such that the cost functional

\[ J(K_f, x_c(0), x_{IM}(0)) = \int_0^\infty e(\tau)^T \Upsilon_c e(\tau) d\tau, \]

\[ \Upsilon_c = \Upsilon'_c \geq 0, \] is minimized along the trajectories of system (2).

The above Assumptions allow to rewrite (6) with respect to a favorable error variable \( \tilde{x} = x - \Pi w, \) that is the difference between \( x(t) \) and its steady-state value guaranteeing perfect tracking, with \( \Pi = [\Pi'_p, \Pi'_c]' = [\Pi'_p, \Pi'_c, \Pi'_1]' \), yielding

\[ e = C x + Q w = C \tilde{x} + (C \Pi + Q) w = C \tilde{x} \]

with the last equality obtained by (4b), then the cost functional (6) can be equivalently rewritten as

\[ J(K_f, x_c(0), x_{IM}(0)) = \int_0^\infty \tilde{x}(\tau)^T \Upsilon_c \tilde{x}(\tau) d\tau, \]

and \( \Upsilon_c = \Upsilon'_c C. \)

Note that alternative variations of the problem above may be equivalently considered. For instance, Problem 1 may be alternatively formulated and solved by assuming that only one of the design parameters \( K_f \) or \( x_c(0) \) and \( x_{IM}(0) \) are at disposal of the user.

3. ON THE SELECTION OF THE PERFORMANCE INDEX

The Transient Optimization problem is formulated in the previous section in terms of a generic cost functional (6). In this section, we motivate a specific choice of such a cost
index, showing first that the proposed performance index is indeed included in the class defined by (6). Towards this end, we introduce two particularization of the previous cost functional. First, we allow for an additional degree of freedom provided by extended dynamics described by equations of the form

\[
\Sigma_{tr} : \begin{cases} 
\dot{x}_{tr} = A_{tr} x_{tr} \\
y_{tr} = C_{tr} x_{tr}
\end{cases}
\]

(9)

which, among Hurwitz matrices \( A_{tr} \), may be arbitrarily assigned defining the auxiliary tracking error \( z_1(t) = e(t) + y_{tr}(t) \) (shaping the output error) in order to define a new cost functional (6). Interestingly, considering dynamic \( \Sigma_{tr} \) as in (9) does not cause any issue with the problem formulation above, since \( \Sigma_{tr} \) may be simply thought as incorporated - in the form of a unreachable but observable subsystem - into the dynamics of the plant defining a new (extended) plant \( \Sigma_P \) such as

\[
\Sigma_P : \begin{cases} 
\dot{x}_p = \tilde{A}_p x_p + \tilde{B}_p u \\
y_p = C_p x_p + Q w
\end{cases}
\]

(10)

where \( \tilde{A}_p = \text{diag}(A_p, A_{tr}) \), \( \tilde{B}_p = [B_p', 0] \), \( C_p = [C_p, I] \). Moreover, the time-derivative of the auxiliary variable \( z_1 \), namely \( z_2 = \dot{z}_1 \), is explicitly considered into the integrand function for the performance index, by considering then

\[
J(K_f, x_c(0), x_{IM}(0)) = \int_0^{\infty} \left( z_1(\tau) \Upsilon_1 z_1(\tau) + z_2(\tau) \Upsilon_2 z_2(\tau) \right) d\tau
\]

(11)

The index (11) can be rewritten as (8) including \( \Sigma_{tr} \) into \( \Sigma_P \) as suggested in (10) and defining \( \dot{x} = \pi - \Pi w \) with \( \pi' = [\pi_p', x_c', \bar{x}_M'] \), \( \Pi = [\Pi_1, 0] \), and \( (A, B, C, P) \) accordingly to (2b), we obtain \( z_1 = \bar{C} \dot{x} \) as in (7) and \( z_2 = \dot{z}_1 = \bar{C} \bar{A} \bar{x} \), considering that, by (4a),

\[
\dot{x} = \bar{A} \dot{x} + \bar{B} w - \bar{P} S w = (-\bar{P} S + \bar{A} \bar{P} + \bar{P} \bar{x} + \bar{A} \bar{x}) = \bar{A} \dot{x}.
\]

Hence, the cost functional in (11) is equivalent to

\[
J(K_f, x_c(0), x_{IM}(0)) = \int_0^{\infty} \dot{x}'(\tau) \Gamma \dot{x}(\tau) d\tau
\]

(12)

\[
\Gamma \equiv [\bar{C}' , \bar{A}' \bar{C}'] \Upsilon = \begin{bmatrix} \Upsilon_1 & 0 \\ 0 & \Upsilon_2 \end{bmatrix}
\]

The rest of this section is devoted to motivate the rationale behind the selection of the cost index (12). Towards this end and to provide an intuitive interpretation, suppose that \( e \in \mathbb{R} \), hence \( z_1 \in \mathbb{R} \) and \( z_2 \in \mathbb{R} \), consider closed-loop error dynamics defined by a second-order system described by equations of the form

\[
\dot{z} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta \omega_n \end{bmatrix} z.
\]

(13)

with \( z = [z_1, z_2]' \), written in terms of the damping factor \( \zeta \) leading to the overshoot \( %S = 100 \exp(-\zeta \pi / \sqrt{1-\zeta^2}) \) when \( 0 < \zeta < 1 \), and the natural frequency \( \omega_n \) that defines a settling time \( T_s \approx -\ln(0.01%)/\zeta \omega_n \) meaning that \( |z_1(t)| < T_s \) for all \( t \geq T_s \). Assuming that the closed-loop dynamics are given by the second order system (13) yields a direct measure of controller performance in terms of classical targets such as overshoot and settling time. Suppose initially that \( y_{tr} \equiv 0 \), then \( z_1 = e \) and \( z_2 = \dot{e} \), and select \( \Upsilon_1 = 1 \) and \( \Upsilon_2 = \beta > 0 \), then the cost index (12) can be rewritten as

\[
J(K_f, x_c(0), x_{IM}(0)) = \int_0^{\infty} \dot{x}'(\tau) \Gamma \dot{x}(\tau) d\tau
\]

It appears evident that the \( L_2 \)-norm \( \|e\|_{L_2} \) is related to the settling time, whereas the relation between the term \( \beta \|\dot{e}\|_{L_2} \) and the overshoot is discussed next. Since the origin of \( \mathbb{R}^2 \) is an asymptotically (exponentially) stable equilibrium point for (13), there exists a unique matrix \( P = P' > 0 \) such that \( A' P + P A = -\Upsilon \). In particular,

\[
P = \begin{bmatrix} 1 & 4 \zeta^2 + 1 + \beta \omega_n^2 & \frac{1}{4} \\ \frac{1}{4} & 1 + \frac{1}{4} + \beta \omega_n^2 & \frac{\omega_n^2}{4} \\ \frac{\omega_n^2}{4} & \frac{\omega_n^2}{4} & \omega_n^2 \end{bmatrix}
\]

(14)

It can be then shown that \( J = z(0)' P z(0) \). As usual, the overshoot is estimated by considering the step response of the system (13) initialized at \( z(0) = [e(0), \dot{e}(0)] = [0, 0] \) and, considering the matrix \( P \) as in (14), it follows that

\[
\frac{\partial J}{\partial \zeta} = \omega_n \left( \frac{1 - \frac{1}{4} + \omega_n^2 \beta}{\zeta^2} \right) e(0)^2.
\]

It can be then concluded that \( \partial J/\partial \zeta \) increases when \( \beta > 0 \) grows (\( \zeta \in (0, 1) \) and for any \( \omega_n > 0 \)). Then, larger values of \( \beta \) are associated to larger penalizing terms for higher overshoots (and consequent oscillations). On the contrary, off-diagonal elements in \( \Upsilon \) lead to increasing \( (\partial J/\partial \zeta) \zeta \), hence are chosen equal to zero. Fig. 2 shows the cost function \( J \) with respect to \( \zeta \) for different values of \( \beta \), with \( z(0) = [e(0), \dot{e}(0)] = [1, 0] \) and \( \omega = 1 \).

We now motivate the role played by the system \( \Sigma_{tr} \) to define the performance output \( z_1 \). Consider the explanatory case in which the (closed-loop) dynamics in (2) with \( K_f = 0 \) and \( Q = 1 \) are described by

\[
\dot{x} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta \omega_n \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w, \quad y = \begin{bmatrix} -\omega_n^2 \end{bmatrix} x, \quad (15a)
\]

and \( w(t) \equiv 1 \) is the reference signal. We have discussed above the importance of selecting \( \beta > 0 \) when overshoots have to be penalized. However, if \( y_{tr} \equiv 0 \), the term \( \beta \|\dot{e}\|_{L_2} \) of \( J \) increases for shorter settling time (or increasing \( \omega_n \)), since the signal \( \dot{e} = -\dot{y} \) is larger during transients and its contribution on \( J \) is weighted by \( \beta \). Even in this explanatory case, \( A_{tr} \), values that minimize \( J \) can not be computed in closed form. However, in order to visualize
respectively, in terms of the matrix gain $K_f$. As a matter of fact, the above (affine) parameterization is achieved here in terms of $\kappa_f \triangleq \text{vec}(K_f)$, as detailed in the following statements.

Lemma 1. Suppose that $\Pi_a$ and $\Gamma_a$ solve (4). Then,

$$\text{vec} \left( \begin{bmatrix} \Pi_a & \Gamma_a \end{bmatrix} \right) = M^{-1} \text{vec} \left( \begin{bmatrix} \hat{P}_a \\ Q \end{bmatrix} \right) + M^{-1} \left( I \otimes \begin{bmatrix} B_a \\ 0 \end{bmatrix} \right) \kappa_f$$

(16)

$$\triangleq \begin{bmatrix} M_{\Pi_a,1} \\ M_{\Gamma_a,1} \end{bmatrix} + \begin{bmatrix} M_{\Pi_a,2} \\ M_{\Gamma_a,2} \end{bmatrix} \kappa_f,$$

with

$$M = \left( S' \otimes \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} - I \otimes \begin{bmatrix} A_a & B_a \\ C_a & 0 \end{bmatrix} \right)$$

(17)

and $\hat{P}_a = [(B_p D_{eq} Q)', (B_a Q)']'$.

A similar parameterization is obtained for $\Pi_{IM}$ in the following result.

Lemma 2. Suppose that $\Pi_{IM}$ solves (5) with $\Gamma_a$ solution of (4). Then,

$$\text{vec}(\Pi_{IM}) = H^{-1} \left( \begin{bmatrix} M_{\Pi_a,1} \\ 0 \end{bmatrix} + \begin{bmatrix} M_{\Gamma_a,1} & M_{\Gamma_a,2} \end{bmatrix} \kappa_f \right)$$

(18)

$$\triangleq M_{\Pi_{IM},1} + M_{\Pi_{IM},2} \kappa_f$$

with $H = \left[ S' \otimes C_{IM} \right]$.

5. OPTIMAL SELECTION OF FEEDFORWARD GAIN AND REGULATOR STATE

The parameterization of the steady-state matrices $\Pi_a$, $\Gamma_a$ and $\Pi_{IM}$ is exploited in this section to provide a solution to Problem 1, or equivalently to its alternative formulation in which only $\kappa_f$ or the initial conditions $x_c(0)$ and $x_{IM}(0)$ are available to the user.

We initially present a solution with respect to the general cost function (6), hence (8), before considering the one introduced and thoroughly motivated in Section 3. To provide a concise statement in the following, note that, for fixed initial condition $w(0)$ of the signal generator, $\Pi_a w(0)$ and $\Pi_{IM} w(0)$ can be rewritten as

$$\Pi_a w(0) = (w(0)' \otimes I) \text{vec}(\Pi_a) \triangleq L_{\Pi_a,1} + L_{\Pi_a,2} \kappa_f$$

$$\Pi_{IM} w(0) = (w(0)' \otimes I) \text{vec}(\Pi_{IM}) \triangleq L_{\Pi_{IM},1} + L_{\Pi_{IM},2} \kappa_f,$$

respectively. Clearly, recalling the partition of $\Pi_a$ in terms of the state $x_p$ and the stabilizer state $x_c$, it follows that $\Pi_a w(0) = L_{\Pi_a,1} + L_{\Pi_a,2} \kappa_f$, with $L_{\Pi_a,i} = [I, 0] L_{\Pi_a,i}$ and $\Pi_{IM} w(0) = L_{\Pi_{IM},1} + L_{\Pi_{IM},2} \kappa_f$, with $L_{\Pi_{IM},i} = [I, 0] L_{\Pi_{IM},i}$, $i = 1, 2$, respectively. Finally, consider the compact notation $x_R = (x_c', x_{IM}')'$ for the entire state of the regulator, comprising the stabilizer and the internal model unit. A similar partition is then obtained also for the matrices $\Pi_a$, $\Pi_{IM}$ and $L_{\Pi_a,i}$, $L_{\Pi_{IM},i}$ in terms of $x_R$ and $L_{\Pi_R,i}$, respectively. The following results are presented to minimize the general cost functional (8) of Problem 1 and can be easily rewritten to minimize (12) with the matrices in Section 3.

Proposition 1. Consider the interconnected system (1d)-(2), under Assumptions 1, 2 and 3, and the cost functional

Remark 2. Should an asymptotically stable filter on the reference signal $w(t)$ be considered in place of $y_{tr}$, to define the performance output $z$, then the former would introduce phase lag and amplitude modification of the reference signal, which is not a desirable feature in the selection of a performance index $J$ that should lead to improved transients. Furthermore, the system $\Sigma_{tr}$ is considered only for optimization purposes, i.e. to define the performance output $z_1$, and is not implemented in the final control scheme.

4. PARAMETERIZATION OF STEADY-STATE MATRICES IN TERMS OF $K_F$

The aim of this brief section consists in parameterizing the matrices $\Pi_a$, $\Gamma_a$ and $\Pi_{IM}$ yielded by (4) and (5), respectively, in terms of the matrix gain $K_f$. As a matter of fact, the above (affine) parameterization is achieved here in terms of $K_f \triangleq \text{vec}(K_f)$, as detailed in the following statements.

Lemma 1. Suppose that $\Pi_a$ and $\Gamma_a$ solve (4). Then,

$$\text{vec} \left( \begin{bmatrix} \Pi_a & \Gamma_a \end{bmatrix} \right) = M^{-1} \text{vec} \left( \begin{bmatrix} \hat{P}_a \\ Q \end{bmatrix} \right) + M^{-1} \left( I \otimes \begin{bmatrix} B_a \\ 0 \end{bmatrix} \right) \kappa_f$$

(16)
Suppose that the state \((x(t), w(t))\) is measured and that \(mq \leq n_p\). Let \(P = P^r > 0\) be such that \(A^TP + PA = \tilde{\Sigma}_e\), and partitioned as \(P = \begin{bmatrix} P_P & P_{PR} \\ P_{PR}^T & P_R \end{bmatrix}\). Then,
\[
\begin{bmatrix} \kappa_f^* \bar{x}_R(0) \\ \bar{x}_R(0) \end{bmatrix} = -\frac{1}{2}P^{-1} \begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \end{bmatrix}
\]
with \(P = \begin{bmatrix} -L_{\Pi_p,2} - L_{\Pi_{R,2}}^T \\ 0 \end{bmatrix} P \begin{bmatrix} -L_{\Pi_p,2} \\ L_{\Pi_{R,2}} \end{bmatrix}^T\),
\[
\bar{b}_1 = (x_p(0) - 2P_{PR}L_{\Pi_{R,2}}) + 2L_{\Pi_{R,1}}P_{PR}L_{\Pi_{R,2}} + L_{\Pi_{R,1}}^TCP_{PR}L_{\Pi_{R,2}}
\]
and \(\bar{b}_2 = (x_p(0) - L_{\Pi_{R,1}})P_{PR} - 2L_{\Pi_{R,1}}^TP_{PR}\). is the unique solution of Problem 1, minimizing the cost \((8)\) along the trajectories of the interconnected system \((1d)-(2)\).

Remark 3. The actual solution to Problem 1 should be obtained by re-shaping (i.e., the inverse operation with respect to \(\text{vec}()\)) the solution \(\kappa_f^*\) provided in \((19)\).

In the two following corollaries, we specialize the minimization results of Proposition 1 to the cases in which only one between \(\kappa_f\) and \(\kappa_f(0)\) is a degree of freedom. Clearly, the conditions become more intuitive in these scenarios. Consider first the case in which \(\kappa_f\) has been fixed.

Corollary 1. Consider the interconnected system \((1d)-(2)\), under Assumptions 1, 2 and 3, and the cost functional \((8)\). Suppose that the state \((x(t), w(t))\) is measured. Let \(P = P^r > 0\) be such that \(A^TP + PA = \tilde{\Sigma}_e\). Then,
\[
x_R(0) = \Pi_R w(0) - P_{PR}^{-1}P_{PR}^T(x_p(0) - \Pi_rw(0))
\]
is the unique solution to Problem 1 for \(\kappa_f\) fixed.

Note that a standard Schur complement argument easily shows that positive definiteness of the matrix \(P\) implies positive definiteness of the matrix \(P_R\), hence the optimal solution can be obtained in this scenario without any condition on the relative dimensions of the plant and of the exosystem and the number of inputs. The structure of the optimal initial condition for the regulator, as a function of \(x_p(0)\) and \(w(0)\), is particularly interesting. In fact, the first term \(\Pi_Rw(0)\), namely with the initial condition of the regulator somewhat coordinated with the initial condition of the exosystem, is typically thought of as the correct initial condition, in the sense that it already satisfies the relation that describes the state of the regulator at steady-state. However, this choice might not be the best in terms of \((8)\) or \((12)\) if the plant initial conditions \(x_p(0)\) are not the ones of the steady-state configuration (that would yield no transient at all if the regulator states were initialized at their steady-state value as well). The following result allows us to additionally quantify the gain on the value of the cost yielded by the initialization \(x_R(0)\) in \((20)\) and the classical \(\bar{x}_R(0) = \Pi_R w(0)\).

Proposition 2. Consider the interconnected system \((1d)-(2)\), under Assumptions 1, Assumption 2 and Assumption 3, and the cost functional \((8)\). Suppose that the state \((x(t), w(t))\) is measured. Let \(P = P^r > 0\) be such that \(A^TP + PA = \tilde{\Sigma}_e\). Then, for fixed \(\kappa_f\),
\[
J(\kappa_f, \bar{x}_R(0)) - J(\kappa_f, x_R^*(0)) = 2(x_p(0) - \Pi_rw(0))^TP_{PR}^{-1}P_{PR}^T(x_p(0) - \Pi_rw(0)).
\]
6. SIMULATIONS

In this section we provide a simple example to show that, even if the regulator state $x_R = [x, x_{IM}]$ is not optimized, the tracking transients can be sensibly improved, i.e. we design only the feedforward matrix $K_f$.

Consider a second order LTI plant with
\[
A_p = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}, \quad B_p = \begin{bmatrix} b \\ 1 \end{bmatrix}, \quad C_p = [a_0, 0], \quad D_p = 0,
\]

with a simple negative unitary (feedback) controller (no states) with $D_{cIM} = -1$, $D_{cp} = 0$ and an integrator $A_{IM} = 0$, $B_{IM} = 1$, $C_{IM} = 1$, and where the exosystem generates a constant reference $w(t) = 1$ ($S = 0$) with the tracking error as $e = C_p x_p - w, \quad Q = -1$. Then
\[
\begin{align*}
A &= \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \quad P &= \begin{bmatrix} b K_f \end{bmatrix} \quad K_f 
\end{align*}
\]

As first, let $y_{tr} \equiv 0$ and consider directly the cost functional (12), then
\[
P = \begin{bmatrix} a_1^2 + \beta (a_1 + a_0^2) & a_1 + \beta & -a_0 \beta + 1 \\ 2(a_0 a_1 - 1) & 0 & 2(a_0 a_1 - 1) \\ \ast & 2(a_0 a_1 - 1) & 0 \\ \ast & \ast & 0 \end{bmatrix},
\]

and the solution of the Francis equations (4) is
\[
\Pi_a = \begin{bmatrix} 1 \\ -a_0 \\ 0 \end{bmatrix}, \quad \Pi_a = \Pi_{IM} = \frac{a_0 - a_1 b K_f - K_f}{a_1 b + 1}
\]

We first optimize (12) only with respect to $K_f$ assuming that $(x_p, w)$ are measurable, hence exploiting the result stated in Corollary 2 replacing $\hat{\gamma}$ with $\hat{\gamma}$.

In the top plot of Fig. 4 we depict different trajectories of $y(t)$: solid blue when no feed-forward is considered ($K_f = 0$), solid black when $K_f = a_0$ is the inverse dc gain of the plant whereas cyan curves are generated with $K_f$ obtained for different values of $\beta$. It is possible to appreciate the reduced number of oscillations for increasing $\beta$. Nevertheless, when $\beta$ increases, since $y_{tr} \equiv 0$, also the convergence speed of $y(t)$ to its asymptotic value is reduced. In the same top plot it is shown how the system output changes picking $\beta = 0.5$ and varying $y_{tr}$ meanwhile $K_f$ is obtained by Corollary 2 with plant redefinition suggested in Section 3. Results agree with the ones presented in Section 3, given that the eigenvalues of $A$ are
\[
\Lambda(A) = \{-0.914 \pm 1.6279j, -0.1721\}.
\]

In the lower plot of Fig. 4 the case $b = -0.2$ is considered, i.e. the plant has an unstable zero. In this case the use of $K_f$ sensibly improves the system performance.

7. CONCLUSIONS

Feedforward design and regulator state resets have been studied to improve the tracking performance in output regulation. Measured and unmeasured plant/exosystem states are considered, and explicit expressions of the optimal feedforward matrix $K_f$ are provided, with a discussion on cost functionals whose minimization yields classical desirable targets such as small settling times and overshoots.

REFERENCES


Huang, J. (2004). Nonlinear Output Regulation. SIAM.


