Hybrid Observer for multi-frequency signals

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Abstract: We proposed a hybrid observer to estimate the frequencies of a signal even in the presence of saturation. Semi-global exponential convergence of the estimation error is provided, and the problem of retrieve dynamically the angular frequencies is addressed.

Keywords: Hybrid observer, frequency estimation, identification, saturation.

1. INTRODUCTION

The problem of estimating the \( n \) unknown angular frequencies \( \omega_i \) of the signal

\[
y(t) = \sum_{i=1}^{n} E_i \sin(\omega_i t + \phi_i),
\]

amplitudes \( E_i \) and phases \( \phi_i \), for \( i = 1, \ldots, n \), has been widely studied in the past given its importance within different scientific fields as identification and control, acoustic, signal analysis, telecommunication. Classic off-line solutions make use of Fourier transform to process sets of batch data (see S.M.Kay and Marple (1981)). Afterwards on-line methods, suitable for many engineering applications, have been firstly proposed in the case of a single frequency signal employing infinite impulse response filter in Regalia (1991) yielding local results, and then combined with adaptation mechanism in Hsu et al. (1999) to yield global results. In Bittanti and Savaresi (2000) a modified extended Kalman filter allows to estimate the frequency of signal in the presence of additive broad-band noise.

In the sequel, just to name a few, global multi-frequency estimator have been proposed in Oregón-Pulido et al. (2002) and Xia (2002) exploiting adaptive identifiers, and in Marino and Tomei (2002), mainly relying on a filtered transformation of co-ordinates, with improved performances. In Marino et al. (2003), the asymptotic estimates of the angular frequencies have been used to cancel out the noise affecting the feedback signals. The amplitudes \( E_i \) have been reconstructed in Hou (2007) via adaptive identifiers.

Within the general framework of Immersion and Invariance observers proposed in Karagiannis et al. (2008), a reduced order observer of dimension \((3n-1)\) has been proposed to solve the same problem in Carnevale and Astolfi (2008), also in the case of a single frequency saturated signal Carnevale and Astolfi (2009).

In this work we propose an hybrid observer, having discrete-time and continuous-time dynamics, which allows to reduce the complexity of the continuous time dynamics of the observers usually proposed to solve this problem, and to solve the case of multifrequency saturated signal, i.e. when the measured signal is of the form

\[
y(t) = \text{sat}_\sigma \left( \sum_{i=1}^{n} E_i \sin(\omega_i t + \phi_i) \right),
\]

where \( \sigma > 0 \) is the saturation level and \( \text{sat}(\cdot) \) is the saturation function defined as

\[
\text{sat}_\sigma(x) = \max(-\sigma, \min(\sigma, x)),
\]

extending the result in Carnevale and Astolfi (2009).

Since the structure we propose exploits sampling of the signals (1) and (2), with a specific sampling time, the results we derive are semi-global given that only signal frequencies lower than half of the sampling frequency can be reconstructed (aliasing). However, in practice, sampling is mandatory and the same limitations applies for the implementation of global observers too.

From a numerical point of view, the algorithm in Section 4.2 supplies estimates of \( \omega_i \)'s with improved transient with respect to the one in Theorem 1 which, as the greater part of the observers devoted to this problem, provides indirect estimate of \( \omega_i \)'s estimating the characteristic polynomial of the LTI system whose output is (1).

2. PRELIMINARIES

To estimate the unknown frequencies \( \omega_i \) of the signal (1), we propose an hybrid observer of the form given in Goebel et al. (2009). Some of the main definitions for this class of hybrid system are recalled next. The reader should refer to Goebel et al. (2009) for further details.
2.1 Notations for hybrid system

Definition 1. A compact hybrid time domain is a set \( T \subset \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \) given by:
\[
T = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j),
\]
where \( J \in \mathbb{R}_{\geq 0} \) and \( 0 = t_0 = t_1 \leq \cdots \leq t_J \). A hybrid time domain is a set \( T \subset \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \) such that, for each \((T, J) \in T, \cap (\{0, T] \times \{0, \ldots, J\})\) is a compact hybrid time domain. \( \square \)

Definition 2. A hybrid trajectory is a pair (dom \( \chi, \chi \)) consisting of a hybrid time domain dom \( \chi \) and a function \( \chi \) defined on \( \chi \) that is continuously differentiable on \((\text{dom} \chi) \cap (\mathbb{R}_{\geq 0} \times \{j\})\) for each \( j \in \mathbb{J}_{\geq 0} \). \( \square \)

Definition 3. For the hybrid system \( H \) given by the open state space \( O \subset \mathbb{R}^n \), and the data \((F, G, C, D)\) where \( F : O \rightarrow \mathbb{R}^n \) is continuous, \( G : O \rightarrow O \) is locally bounded, and \( C \) and \( D \) are subsets of \( O \), a hybrid arc \( \chi : dom \chi \rightarrow O \) is a solution to the hybrid system \( H \) if \( \chi(0, 0) \in C \cup D \) and the following hold.

(1) For all \( j \in \mathbb{J}_{\geq 0} \), and for almost all \( t \in I_j := \text{dom} \chi \cap (\mathbb{R}_{\geq 0} \times \{j\}) \), we have \( \chi(t, j) \in C \) and \( \dot{\chi}(t, j) = F(\chi(t, j)) \).

(2) For all \((t, j) \in \text{dom} \chi \), such that \((t, j + 1) \in \text{dom} \chi \), we have \( \chi(t, j + 1) = G(\chi(t, j)) \) with \( \dot{\chi}(t, j + 1) \in D \). \( \square \)

Hence, the hybrid system model that we consider is of the form:
\[
\dot{\chi}(t, j) = F(\chi(t, j)) \quad \chi(t, j) \in C,
\]
\[
\chi(t, j + 1) = G(\chi(t, j + 1)) \quad \chi(t, j + 1) \in D.
\]
\( F() \) and \( G() \) are usually called flow map and jump map, respectively. In the sequel (as in Goebel and Teel (2006)) we omit the time arguments when possible and write:
\[
\dot{\chi} = F(\chi) \quad \chi \in C,
\]
\[
\chi^+ = G(\chi) \quad \chi \in D,
\]
where we denoted \( \chi(t, j + 1) \) as \( \chi^+ \) in the last equation.

The next definitions are needed to state the observer properties.

Definition 4. A hybrid arc \((\chi(t, j))\) uniformly converges to a set \( A \) if there exists a function \( \beta \in KLL \) such that
\[
|\chi(t, j)|_A \leq \beta(|\chi(0, 0)|_A, t, j),
\]
for all \((t, j) \in \text{dom} \chi \), where \( |s|_A = \inf_{a \in A} ||s - a||. \) \((\chi(t, j))\) uniformly globally exponentially converges to the set \( A \) if \( \beta \) can be taken to have the form \( \beta(s, t, j) = Ms \exp(-\lambda(t + j)) \) for some \( M > 0 \) and \( \lambda > 0 \). \( \square \)

2.2 Signal generator

We assume that the signal (1) is the output \( y(t) = Cx(t), x \in \mathbb{R}^{2^n}, C_{1 \times 2^n} = [0, 1, 0, 1, 0, \ldots, 1], \) of the linear time invariant system
\[
\dot{x} = Ax = \text{diag} \left\{ \begin{array}{c} 0 \\ \omega_i \\ 0 \end{array} \right\} x, \quad i = 1, \ldots, n, \quad (4)
\]
with unknown initial condition \( x(0) = x(t_0) \). We also assume that samples \( y(t_k) \) of \( y(t) \) are available with
\[
\text{sampling time} T \text{ such that } t_k = t_{k-1} = T \text{ for each } k \in \mathbb{N} \text{ and } t_0 = 0. \text{ Due to sampling, the relations}
\]
\[
x(t_{k+1}) = AD x(t_k), \quad y(t_k) = Cx(t_k)
\]
hold where
\[
A_D := e^{AT} \quad (5)
\]
with \( A_D \) having characteristic polynomial
\[
p_{AD}(\lambda) = (\lambda^2 - 2\cos(\omega_1)\lambda + 1) \cdots (\lambda^2 - 2\cos(\omega_n)\lambda + 1) = \lambda^{2n} + a_{2n-1} \lambda^{2n-1} + \cdots + a_1 \lambda + a_0 \quad (6)
\]
having symmetric coefficients, i.e. such that \( a_{2n-k} = a_h \), \( h = 1, 2, \ldots, n - 1 \). By the previous property, the coefficients of \( p_{AD}(\lambda) \) can be compactly expressed as
\[
a := \begin{bmatrix} 1 \\ a_1 \\ a_2 \\ \vdots \\ a_{2n-2} \\ a_{2n-1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} S \begin{bmatrix} 1 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} = \hat{S} \begin{bmatrix} 1 \\ a_c \end{bmatrix}, \quad (7a)
\]
where, denoting by \( e_i \) the \( i \)-th column of the identity matrix and by \( S_i \) the \( i \)-th row of \( S \in \mathbb{R}^{2n-1 \times n} \), the matrices \( S, \hat{S} \) and the compact coefficient vector \( a_c \) are defined according to
\[
S_i := \begin{cases} 1 & \text{if } i = j \text{ or } 2n - i = j, \\ 0 & \text{otherwise}, \end{cases} \quad (8a)
\]
\[
\hat{S} = \text{blockdiag}(1, S), \quad (8b)
\]
\[
a_c = \begin{cases} 1 & \text{if } i = 1, \ldots, n, \end{cases} \quad (8c)
\]

Our objective is to obtain either estimates \( \hat{\omega}_i \) of \( \omega_i, i = 1, \ldots, n \), or estimates \( \hat{a}_i \) of the coefficients \( a_i, i = 1, \ldots, n \). Note that, from the estimate \( \hat{\omega}_i \), an estimate \( \hat{a}_i \) is immediately obtained via the relation \( \hat{a}_i = \hat{S}_a e_i \), with \( \hat{a} \) containing the coefficients of the characteristic polynomial
\[
p_{AD}(\lambda) = \lambda^{2n} + \hat{a}_{2n-1} \lambda^{2n-1} + \cdots + \hat{a}_1 \lambda + \hat{a}_0, \quad (9)
\]
of matrix
\[
\hat{A}_D := e^{AT}, \quad \hat{A} := \text{diag} \left\{ \begin{array}{c} 0 \\ -\hat{\omega}_i \\ 0 \end{array} \right\}, \quad i = 1, \ldots, n. \quad (10)
\]

Note that both \( p_{AD}(\lambda) \) and \( p_{AD}(\hat{\lambda}) \) have all roots of modulus equal to one (since their coefficients are symmetric). Moreover, the relation between the \( \omega \) and \( \hat{a}_c \) is readily obtained considering that
\[
p_{AD}(\lambda) = \prod_{i=1}^{n} (\lambda^2 - 2\cos(\omega_i T)\lambda + 1),
\]
and is expressed by a function \( \hat{a}_c = f(\omega) \) such that each component \( f_h(\omega), h = 1, \ldots, n \), contains a sum of monomials in \( \cos(\omega_i T), i = 1, \ldots, n \), of degree not larger than \( h \) and where each \( \cos(\omega_i T) \) appears with exponent equal to either 1 or 0. Clearly, the same relation holds between \( \hat{\omega} \) and \( \hat{a}_c \), namely \( \hat{a}_c = f(\hat{\omega}) \).

Define
\[
O := \begin{bmatrix} C \\ CA_D \\ \vdots \end{bmatrix}, \quad \hat{O} := \begin{bmatrix} C \\ CA_D \hat{A} \\ \vdots \end{bmatrix}, \quad (11)
\]

\(2\) The value of \( T \) can be selected as an integer multiple of the sampling time of the ADC.
Lemma 2.\[x \quad Y \quad O \quad T \quad with conjugate (not real) eigenvalues, and then the fact that\]

Define \( \tilde{\sigma} = \tilde{\sigma}(t) = \sigma(t) \) and \( Y_k := \begin{bmatrix} y(t_{k-2n}) \\ \vdots \\ y(t_{k-1}) \end{bmatrix} = Ox(t_{k-2n}), \) (13)

so that \( x(t_{k-2n}) = O^{-1}Y_k; \) then for \( t \in (t_k, t_{k+1}) \) the error between \( y(t_k) \) and its estimated value \( \hat{y}(t) \) based on the measures in \( Y_k \) and the estimated parameters \( \tilde{a}(t) \) is given by \( e(t) = \tilde{a}(t)Y_k \)

\[ e(t) := y(t_k) - \hat{y}(t) \]

\[ e(t) = (O^{-1} - (t)O)x(t_{k-2n}) \]

\[ e(t) := (O^{-1} - (t)O)x(t_{k-2n}) = \tilde{a}(t)Y_k \] (14)

(14a)

(14b)

(14c)

(14d)

Note that an initial state \( x(t_0) \) excites all the modes if at least one between \( x_{2n+1}(t_0) \) and \( x_{2n+2}(t_0) \) is nonzero, for all \( i = 0, 1, \ldots, n - 1 \). Further, define for \( k \geq 4n\)

\[ X_k := [x(t_{k-4n+1}) \quad x(t_{k-4n+2}) \cdots x(t_{k-2n} \cdots x(t_{k-2n})], \]

\[ Y_k := [Y_{k-2n+1} \\ Y_{k-2n+2} \cdots Y_{k-1} Y_{k-1}] \]

Lemma 1. Let \( x(t_0) \) be such that all modes are excited. For almost any choice of \( T \), and for all \( k \geq 4n \), the matrices \( X_k \) and \( Y_k \) have full rank.

Proof. We proceed by induction, and prove first the claim about \( X_k \).

First, consider the null case \( k = 4n \). For almost all choices of \( T \), matrix \( A_D = e^{AT} \) has exactly \( n \) distinct pairs of complex conjugate (not real) eigenvalues, and then the fact that \( x(t_0) \) excites all the modes imply that the pair \( (A_D, B_0) \) with \( B_0 := x(t_0) \) is reachable; hence the reachability matrix of \( (A_D, B_0) \), coinciding with \( X_k \) for \( k = 4n \), is full rank.

To prove that \( x(t_{k+1}) = 2n \) implies \( \text{rank}(X_{k+1}) = 2n \), note that \( X_{k+1} = A_DX_k \) and \( A_D = e^{AT} \) is nonsingular.

To verify the claim about the matrices \( Y_k \), note that by the relations \( Y_k = Ox(t_{k-2n}) \), \( k = 2n + 1, 2n + 2, \ldots, k \), it follows that \( Y_k = OX_k \) and then \( \text{rank}(Y_k) = 2n \) since \( O \) is invertible for almost all \( T \) and \( \text{rank}(X_k) = 2n \).

Lemma 2. \( \|x(t_k)\| \leq \|x(t_0)\| \) for all \( k \geq 0 \). Moreover, for almost all \( T \) \( \|Y_k\| \geq \sigma(O) \|x(t_0)\| \) for all \( k \geq 4n \).

Proof. Since \( x(t_{k+1}) = A_Dx(t_k) \) and \( A_D^2 = A_D^{-1} \), then \( x(t_{k+1}) = x(t_k)A_Dx(t_k) = x(t_k)x(t_k) \), so that \( \|x(t_k)\|^2 = x(t_k)x(t_k) = \|x(t_k)\|^2 \) for all \( k \geq 4n \). Since \( O \) is invertible then \( \sigma(O) > 0 \) and since \( Y_k = Ox(t_{2n-2n}) \) then \( \|Y_k\| \geq \sigma(O) \|x(t_0)\| \).

3. Estimating the Characteristic Polynomial

3.1 Static estimation

Note that the characteristic polynomial coefficients \( a \) can be directly evaluated exploiting the fact that \( Y_{k+1} = \begin{bmatrix} y(t_{k-2n+1}) \\ \vdots \\ y(t_{k-1}) \\ y(t_k) \end{bmatrix} = \begin{bmatrix} y(t_{k-4n+1}) \cdots y(t_{k-2n}) \\ \vdots \\ y(t_{k-2n+1}) \cdots y(t_{k-2n}) \end{bmatrix} a \)

\[ \text{and Lemma 1 yielding} \]

\[ a = -Y_k^{-1}Y_{k+1} \] (17)

In this case, before obtaining the exact value of \( a \), we have to collect \( 4n \) data, consequently, we have to wait \( 4T \) time units. The number of measures can be decreased down to \( 3n \) using the definition of \( a_c \) in (7) which yields \( Y_k = Y_{k-2n} = -Y_{k-1}S\alpha \), where

\[ Y_k^c := \begin{bmatrix} y(t_{k-2n+1}) \\ \vdots \\ y(t_k) \end{bmatrix} \in \mathbb{R}^n, \]

\[ Y_{k-1}^c := \begin{bmatrix} y(t_{k-2n+1})^c \\ \vdots \\ Y_{k-1} \end{bmatrix} \in \mathbb{R}^{n \times 2n - 1} \] (18a)

(18b)

then \( a_c \) can be evaluated as

\[ a_c = -\left( S'Y_k \right)^{-1} \left( S'Y_{k-1} \right) (Y_k^c - Y_{k-2n}^c). \] (19)

The invertibility of the matrix \( S'Y_k \) is ensured by \( \text{rank}(Y_k) = n \) for any \( n \geq 3n \), that directly follows from the definition of \( S \), that can be partitioned as \( S' = \left[ I_{n \times n}, 0 \right]^T \), and the \( n \) independent vectors in \( Y_k^c \) (see Lemma 1).

Straightforward inversion as (19) are strongly affected by noise on the measures, which may even lead matrices \( Y_k^c \) and \( S'Y_k \) to singularity. In this case, to mitigate the corruption of the estimate, it would be possible to consider the classical moving window technique, in which a number of measures larger than \( 4n \) (3n) are lined up defining a “taller” \( Y_k \) and \( Y_k^c \). This well known technique reduces the effect of the noise as shown in Figure 1.

Note that the \( E_i \)'s and \( \phi_i \)'s can be evaluated directly by \( x_{k-2n} = O^{-1}Y_k \).

3.2 Dynamic estimation

Let

\[ A_0 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 \\ \vdots \\ \vdots \end{bmatrix}, \quad (20) \]

with \( A_0 \in \mathbb{R}^{2n \times 2n} \) and \( B_0 \in \mathbb{R}^{2n} \). Define the hybrid observer \( H \) with state

\[ \xi = \begin{bmatrix} z_c \quad \zeta' \quad \tau' \end{bmatrix} \]

\[ \text{with } \zeta, \zeta' \in \mathbb{R}^n, \quad \tau, \tau' \in \mathbb{R}, \quad \text{having dynamics defined in terms of the flow and jump maps} \]
\[
\dot{\hat{a}}_c = -\gamma S'\zeta e, \\
\dot{\zeta} = 0, \\
\dot{\chi} = 0, \\
\dot{\tau} = 1, \\
\dot{\zeta}^+ = \dot{a}_c, \\
\zeta^+ = \zeta_0 + \zeta, \\
\chi^+ = \chi, \\
\tau^+ = 0, \\
\text{if } \xi \in \mathcal{C}, \\
\text{if } \xi \in \mathcal{D}, \\
\text{if } \xi \in \mathcal{F},
\]
(22)

where \(S' = [0, S'] \in \mathbb{R}^{n \times 2n}, \gamma > 0\) is the observer gain, and
\[
e = \chi - \zeta' [1, a'_c]' = y(tk) - Y_k S'[1, a'_c]',
\]
The flow set \(\mathcal{C}\) and the jump set \(\mathcal{D}\) are defined as
\[
\mathcal{C} \triangleq \{ \xi : \tau \in [0, T] \}, \\
\mathcal{D} \triangleq \{ \xi : \tau \geq T \}.
\]
(23a)
(23b)
The variable \(\zeta\) acts as a buffer to maintain the past 2n values of the input \(y\), i.e., \(Y_k = \zeta(k, t)\), whereas \(\chi\) is the last measured value of \(y\), i.e., \(\chi(k, t) = y(tk)\) for all \(t \in [t_k, t_{k+1})\). To implement the observer (22), we consider a sampling frequency of the signal \(y(t)\) which satisfies the Nyquist criteria.

**Assumption 1.** The angular frequencies \(\omega_i\) in (4) are such that \(0 < \omega_i < \pi/T\) for all \(i = 1, \ldots, n\).

The continuous-time dynamics of \(\hat{a}_c\) in (22) have been selected considering the gradient algorithm to minimize the function \(V_c(k, t) = ||\hat{a}_c(k, t)||^2/2\), yielding
\[
\dot{V}_c(k, t) = -\gamma \hat{a}'_c(k, t)S'Y_k Y'_k S'\hat{a}_c(k, t), \\
\text{given that } \zeta(k, t) = Y_k, \text{ and } e(k, t) = y(kT) - \hat{y}(kT) = \chi + \zeta' S'\hat{a}_c(k, t). \\
\]
To claim uniform exponential convergence to zero of the estimation error \(\hat{a}_c(k, t)\) we need \(y_k\) to be persistently exciting (see Sastry and Bodson (1989)).

**Definition 5.** The signal \(Y_k\) is persistently exciting (PE) if there exist \(\gamma_1\) and \(\gamma_2\) such that
\[
0 < \gamma_1 I \leq \sum_{h=0}^{2n-1} Y_{k+h}Y'_{k+h} \leq \gamma_2 I,
\]
(25)
for any \(k \geq 2n\).

Note that once the Assumption 1 holds, the signal \(Y_k\) is PE by Lemma 1.

**Theorem 1.** Under Assumption 1, the estimation error \(\hat{a}_c(k, t) = \hat{a}_c(k, t) - a_c\) uniformly exponentially converges to zero as \(k\) goes to infinity.

The next corollary easily follows from Theorem 1 noting that
\[
e = y(tk) - \hat{y} = Y'_k (S[1, \hat{a}'_c] + a(k, t) - a), \quad Y_k \begin{bmatrix} 0 \\ S\hat{a}_c \end{bmatrix} = \zeta' \begin{bmatrix} 0 \\ S\hat{a}_c \end{bmatrix}.
\]
**Corollary 1.** Under Assumption 1, the estimation error \(\hat{a}_c(k, t) = \hat{a}(k, t) - a\) uniformly exponentially converges to zero as \(k\) goes to infinity.

The proof, not shown here due to space constraint, is based on standard techniques for linear systems exploiting persistence excitation properties. To succeed with the estimation of \(a\) in case of saturation, namely with the signal (2), we need a further assumption which guarantees that we are able to capture an infinite batch of at least 2n consecutive samples within the saturation thresholds.

**Assumption 2.** It holds that
\[
\sum_{s \in \mathcal{U}} t_{s+2n} - t_s = \infty,
\]
(26)
where \(t_s = sT\) and \(s \in \mathcal{U} := \{ s \in \mathbb{N} : |y((s+h)T)| < \sigma, \forall h = 0, 1, \ldots, 2n \}.

Note that the approach pursued in Carnevale and Astolfi (2009) is not easily practicable since the update map of the estimation error depends on the unknown \(\omega_i\), which greatly complicates the design of the observer. However, if we only modify the flow and jump map of the observer (22) as
\[
\hat{a}_c = -\gamma(q) S'\zeta e, \\
\zeta = 0, \\
\chi = 0, \\
\tau = 1, \\
\hat{c} = \hat{c}, \\
\zeta^+ = \zeta_0 + \zeta, \\
\chi^+ = \chi, \\
\tau^+ = 0, \\
\text{if } \xi \in \mathcal{C}, \\
\text{if } \xi \in \mathcal{D}, \\
\text{if } \xi \in \mathcal{F},
\]
(27)

where \(\gamma(q) = \begin{cases} \gamma_s & \text{if } q \geq 2n \\ 0 & \text{otherwise} \end{cases}\)
(28)
\[
G(q) = \begin{cases} q + 1 & \text{if } |\chi| < \sigma \\ 0 & \text{otherwise} \end{cases},
\]
(29)
such that \(q\) counts the number of 2n consecutive samples within the saturation level \(\sigma\), and \(\hat{a}_c\) is “frozen” up to a new \(Y_k\) filled with data \(|y(\tau T)| < \sigma\) is collected.

**Theorem 2.** Under Assumption 1 and Assumption 2, the estimation error \(\hat{a}_c(k, \tau) = \hat{a}_c(k, \tau) - a_c\), with flow and jump maps as in (27), respectively, uniformly exponentially converges to zero as \(k\) goes to infinity.

### 4. ESTIMATING THE FREQUENCIES

The methods in Section 4 yield an estimate of the coefficients of the characteristic polynomial. If it is desired to estimate the value of the coefficients \(\omega_i, i = 1, \ldots, n\), at least three option exist:

1. directly compute the roots of \(p_{\hat{A}_0}(\lambda)\) in (9);
2. build an observer which computes estimates \(\hat{\omega}_i\) of \(\omega_i;\)
3. use an observer for the dynamic inversion of the nonlinear map between the \(\hat{\omega}_i\)'s and the \(a_i\)'s.

These approaches are now briefly described and commented upon.

#### 4.1 Direct computation from \(p_{\hat{A}_0}(\lambda)\)

Any root finding algorithm can be used for finding the roots \(\lambda_i, i = 1, \ldots, 2n, \) of \(p_{\hat{A}_0}(\lambda)\), which come in complex conjugate pairs, i.e., \(\lambda_i = \lambda^*_i + n_i, i = 1, \ldots, n\) from which it is easy to reconstruct the values of the \(\omega_i, i = 1, \ldots, n\), which are related according to \(\lambda_i = e^{\omega_i T}, i = 1, \ldots, n\). Here it is only stressed that the coefficients of \(p_{\hat{A}_0}(\lambda)\) have a symmetric structure which is the same structure of the coefficient of \(p_{\hat{A}_0}(\lambda)\) and is enforced by estimating only \(\hat{a}_c\) and relation (7); hence, the roots of \(p_{\hat{A}_0}(\lambda)\) belong to the unit circle and can be found by a line search with \(\lambda \in (0, \tau T)\) for the function \(p_{\hat{A}_0}(e^{\lambda T})\).

#### 4.2 Dynamic estimation of the \(\omega_i\)'s via an observer

Since \(\hat{a}_c\) can be expressed as a function of \(\hat{\omega}\) according to the relation \(\hat{a}_c = f(\hat{\omega})\), it is possible to replace the dynamics of \(\hat{a}_c\) in (22) with the following dynamics for \(\hat{\omega}\):
\[ \dot{\omega} = -\gamma \nabla f(\omega)^\top \tilde{S}^\top \zeta e, \tag{30a} \]
\[ \dot{\omega}^+ = \dot{\omega}. \tag{30b} \]

Remark 1. The solution in (30) requires the invertibility of matrix \( \nabla f(\omega) \), which can be shown to hold if and only if all the elements of \( \dot{\omega} \) are pairwise different and none of them is zero. Such a condition can be enforced by modifying the right hand side of (30a) by introducing a barrier function \( \gamma \nabla f(\omega) \), with \( \gamma \) and \( \nu \) uniformly distributed pseudorandom numbers.

In the first example, the performances of the static estimator in Assumption 1, namely \( \hat{a}_c = f(\omega) \), are shown with \( d = 0 \) and \( d = 0.05 \), selecting \( T = 0.3 \), \( \gamma = 30 \) and initial conditions \( \hat{\omega}_1(0,0) = \hat{\omega}_2(0,0) = 0 \).

4.3 Dynamic inversion of \( a_c = f(\omega) \) via an observer

An alternative approach is obtained by using the original observer to produce the estimates \( \hat{a}_c \) of \( a_c \), meanwhile using another observer for the inversion of the relation \( \hat{a}_c = f(\omega) \) similar to what is done in Nicosia et al. (1992). This additional observer can be defined via the dynamics:

\[ \dot{\omega} = -\gamma \nabla f(\omega)^\top (\hat{a}_c - f(\omega)), \tag{33} \]
\[ \dot{\omega}^+ = \beta_1(\omega) \tag{34} \]

where the function \( \beta_1(\omega) \) is defined as the solution of the following minimization problem:

\[ \beta_1(\omega) = \arg\min_v \| v - \hat{\omega} \| \tag{35a} \]
\[ \text{s.t. } 0 < v < \frac{\pi}{T}, \tag{35b} \]
\[ v_{i+1} \geq v_i + \epsilon, \quad i = 1, \ldots, n-1, \tag{35c} \]

and \( \hat{\omega} \) is initialized so that \( 0 < \hat{\omega}_1(t_0) < \hat{\omega}_2(t_0) < \cdots < \hat{\omega}_n(t_0) < \frac{\pi}{T}. \)

5. SIMULATION EXAMPLES

In the next examples, we consider the signal \( y(t) = \sin(5t) + \sin(2t) + \sigma \nu(t) \) with \( T \) satisfying the Nyquist criteria in Assumption 1, namely \( T < \pi/5 \).

In the first example, the performances of the static estimation in Section 3.1 using formula (17), \( T = 0.3s \), and uniformly distributed pseudorandom numbers \( \nu(t) \) between 0 and 1 (\texttt{rand} function in Matlab), with \( d = 0.05 \), are shown in Figure 1. The angular frequencies are evaluated analytically from the coefficient vector \( a \).

In the second numerical example we consider the observer in Theorem 1. Note that the observer (22) within the sampling interval becomes simply

\[ \dot{\hat{a}}_c = -\gamma S^T Y_k \epsilon(kt), \tag{36} \]

where \( e(kt) = y(kt) - Y_k \hat{a}_c \), and at the sampling time, the new sample \( y(kt) \) is inserted at the bottom of the vector \( Y_k \), discarding its old first element. This greatly simplifies the implementation of the observer with respect to the previous ones. Note that, within the “real” sampling intervals \( T \), the dynamics of (36) have been simulated for a “virtual” time of \( T_v = 30 \) seconds to improve the estimates. When (36) have to be implemented on-line, the only limitation on \( T_v \) is the hardware speed to evaluate \( \hat{a}_c(k; T_v) \) from the initial condition \( \hat{a}_c(k; k_0) \).

In Figure 2 the numerical results of the observer (36) are shown with \( d = 0 \) and \( d = 0.05 \), selecting \( T = 0.3 \), \( \gamma = 30 \) and initial conditions \( \hat{\omega}_1(0,0) = \hat{\omega}_2(0,0) = 0 \).

We have experienced that the key parameter to improve performances of the algorithm is \( T \), provided that you can ensure a “virtual” time \( T_v \) much greater.

The good choice of \( T \) may also be evaluated on-line, checking the conditioning number of \( Y_k \) defined in Section 3.1. The observer (36) is then very simple in his mathematical
expression, easily tunable and quickly computable. There is still the problem of evaluating \( \omega_i \)'s from \( a \), but it can be addressed with the tools provided in Section 4.

To improve the transient, we show the simulation example in Figure 3 of the observer (32)-(30b), which supplies directly the \( \hat{\omega}_i \)'s. The simulation has been performed with \( d = 0 \) and \( \gamma = 0.05 \), selecting \( T = 0.3 \), \( \gamma = 30 \), \( \varepsilon = 0.1 \) and initial conditions \( \hat{\omega}_1(0,0) = 1 \) and \( \hat{\omega}_2(0,0) = 0.5 \) \((T_v = 30)\).

![Fig. 3. Dynamic estimation, with and without noise, of the simple observer with dynamics (30b)-(32).](image1)

In the last simulation of Figure 4, we show the performance of the observer in Theorem 2 with \( \sigma = 1.5 \), \( d = 0 \) and \( d = 0.05 \), and selecting \( T = 0.3 \), \( \gamma = 30 \), \( \varepsilon = 0.1 \) and initial conditions \( \hat{\omega}_1(0,0) = 1 \) and \( \hat{\omega}_2(0,0) = 0.5 \) \((T_v = 30)\).

![Fig. 4. Dynamic estimation, with and without noise, of the observer in Theorem 2 with saturation level \( \sigma = 1.5 \).](image2)

6. CONCLUSIONS

We have proposed a semi-global hybrid observer to estimate the angular frequencies of the signal (1) also in the presence of saturation, as in (2). Static (finite time) estimates of \( \omega_i \)'s are compared via simulation results with the proposed dynamic observers. In particular, the observer (30a)-(30b) overcomes the problem treated in Section 4 to retrieve the \( \omega_i \)'s from the coefficients \( a \) of the characteristic polynomial \( p_A(\lambda) \). The simple analytic form and tuning of the proposed observers may be a suitable feature for practical implementations.

REFERENCES


