A Robust Anti-Windup Scheme for Manual Flight Control of an Unstable Aircraft

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Abstract
A result on robust in the large, anti-windup compensation obtained for the case of asymptotically stable systems is extended to deal with an unstable fighter aircraft model. Simulations show the effectiveness of the proposed technique.

1. Introduction
Modern high performance fighter aircraft are characterized by an extended flight envelope, in order to allow the pilot to reach unprecedented maneuvering capabilities at high angles of attack. Such a result can be achieved only exploiting the control power of aerodynamic surfaces as much as possible and this has serious consequences on the challenges for the control engineer. In fact, during aggressive maneuvering, the aerodynamic controls can reach their saturation limit, or, in other cases, the pilot control demand can change abruptly, and rate saturation can affect the overall performance of the control system. Moreover, the coupling of control (magnitude and rate) saturation with the response of a human pilot can be the root for pilot induced oscillations (PIO), that were recognized as the cause of several accidents or critical events (see e.g. [13], [1]).

In many cases instabilities are avoided using so-called flight envelope protection systems, i.e. the pilot input is limited (or even canceled) when dangerous zones of the flight envelope are approached. An example in the publicly available literature is the control system of the F–16 fighter aircraft, where it is possible to observe several of such devices, such as the Automatic Spin Prevention System (ASPS), or the anti-stall system [6]. In both cases pilot commands to rudder and elevator are progressively faded when the angle of attack exceeds a value of 20 deg, until they are cancelled for \( \alpha > 25 \) deg. The parameters of the flight envelope protection system are usually tuned during the flight tests, when the test pilot recognizes a tendency to control loss or a serious degradation of handling qualities.

Of course, the use of flight envelope protection systems prevents the possibility of achieving the actual performance limits of the aircraft, inasmuch a sizable safety margin is usually necessary to avoid that the unsafe region is reached during highly aggressive maneuvering. The problem is further complicated by the instability of the unaugmented dynamics of modern aircraft, so that in presence of control saturation it is not possible for the stability augmentation system (SAS), that artificially provides the necessary stability margin, to avoid departures from controlled flight.

These considerations are at the base of an increasing interest for anti-windup control schemes, where the control system is able to cope with maneuvering segments during which the effectors reach their limits, in terms of position and/or rate saturation. Though several techniques for control in the presence of saturation are available (see e.g. [11], [8], [5] and references therein), the peculiarity of anti-windup control [15] consists in being an “add-on” which can be placed on an arbitrary a-priori given “nominal” high performance compensator (designed without taking saturation into account, e.g. by linear \( H_\infty \) techniques), and such that 1) the response induced by the nominal compensator is not modified until saturation occurs and 2) instabilities due to input saturation during transients are avoided (i.e. any command which is feasible at steady state is asymptotically tracked). In order to accomplish the above task in the case of open loop unstable systems [14], the anti-windup controller must guarantee the additional property that 3) the state of the plant never leaves the null controllable region (i.e. the set of states that can be driven to zero with bounded inputs, which is a strict subset of the state-space in the open loop unstable case). In the case of the F–16, an elementary anti-windup scheme was implemented, in order to limit the risk of integrator windup in the PI element of the longitudinal Stability and Control Augmentation System where the output of the integrator is bounded to a value equal to the stabilator deflection limit and the input is reduced by an amount proportional to the saturation violation of the stabilator command [6]. This scheme strictly satisfies only requirement (1), providing a heuristic, although reasonable, approach for the fulfillment of requirements (2) and (3).

In a recent paper [3], the approach in [14] was extended to account for rate (in addition to magnitude) saturation in the synthesis of an anti-windup compensator, and the special case of manual flight control of an exponentially unstable aircraft (the Tailless Advanced Fighter Aircraft [TATA]) was considered in [2]. In the latter paper, a very effective anti-windup compensator is designed considering a trim condition characterized by a dynamic pressure of 450 psf. The present paper aims at complementing [2] by studying the issue of robustness with respect to variations of trim condition.

In fact, the accuracy of the model is of paramount impor-
tance for the results in [2], inasmuch as when an uncertain model is dealt with, the performance of the anti-windup compensator are no longer guaranteed; more precisely, though performance deterioration can be shown to be arbitrarily small for sufficiently small uncertainties [15], the presence of “large” uncertainties (combined with the anti-windup requirement (1) above) can spark instabilities even in cases when the nominal controller was robust to the same uncertainties [7]; such an observation motivated the introduction of a “weakened” anti-windup problem in [7], where requirement (1) is relaxed in order to allow a robustification of the anti-windup closed loop with respect to larger uncertainties. Clearly, the issue of robustness is crucial, when dealing with aircraft capable of reaching high angles of attack, were nonlinear and/or unsteady aerodynamic effects significantly affect the behavior of the vehicle. As an example, Fig. 1 shows the real and imaginary parts of the eigenvalues for the reduced short-period longitudinal model of the TAFA, in a set of trim conditions that ranges between 170 and 600 knots of equivalent airspeed, that is a dynamic pressure between 100 and 1200 psf.

Such significant changes of the aircraft dynamic behavior usually require the use of gain scheduling, for adapting the SAS to changes in stability derivatives, but this may not be possible when robust MIMO controllers are synthesized using modern techniques, like $\mu-$synthesis. In this case the structure of the controller can change for different operating points, and “bumpless” transfer techniques are needed for managing the switching of the control authority between different controllers. In these circumstances, it is useful to extend as much as possible the operating envelope of each controller, in order to keep the number of switching thresholds as low as possible; and the capability of the anti-windup compensator to work far from the nominal operating point, possibly with performance close to the performance of the nominal compensator, is a very important property.

Considering only magnitude saturation for simplicity, at a given trim condition with dynamic pressure $Q$ the null controllable region for the simplified TAFA model in [2] is a given trim condition with dynamic pressure $Q$ an approximation $V_0$ (dashed) for $Q$ in the range 100-1200 psf.

Fig. 1. Plant eigenvalues as a function of equivalent airspeed.

Fig. 2. Safe region $V$, an approximation $V_0$ (dashed) and a domain of attraction $\mathcal{X}$ (dash-dotted) for $Q$ in the range 100-1200 psf.

dooming any recovery attempt. It is easy to see that such a problem remains even if not only the nominal controller, but also the anti-windup compensator is gain scheduled; in fact, in order to actually guarantee recoverability independently of $Q$, it must be guaranteed that the state never leaves the (much smaller, and of course not infinite!) “safe region” $V = \bigcap_{Q \in [100,1200]} V_Q$ represented in Fig. 2.

In this paper, considering only magnitude saturation, for the first time the approach proposed in [7] for the robustification of anti-windup compensators is extended to unstable plants and is applied in the framework of manual flight control. The proposed compensator solves a suitably weakened anti-windup problem for the reduced short-period longitudinal model of the TAFA, successfully coping with plant dynamics variations and keeping the state inside the safe region $V$, without any scheduling in the considered range of variation of dynamic pressure (100 to 1200 psf).

After the introduction of some notations at the end of this section, Section 2 discusses the extension of the weakened anti-windup problem needed for the control of the TAFA model, Section 3 describes the TAFA model, the design of the anti-windup compensator and shows some simulations, and finally Section 4 provides some conclusions.

Notation

For a given convex set $\mathcal{U} \subset \mathbb{R}^p$ and a vector $u \in \mathbb{R}^p$, let $\text{dist}_{\mathcal{U}}(u) := \inf_{w \in \mathcal{U}} \| u - w \|$, where $\| \cdot \|$ represents the Euclidean norm; $\text{int}\mathcal{U}$ is the interior of $\mathcal{U}$, and the function $\Gamma_{\mathcal{U}}$ is a projection function on $\mathcal{U}$, i.e. $\Gamma_{\mathcal{U}}(v) \in \mathcal{U}$, $\forall v \in \mathbb{R}^p$, and $\Gamma_{\mathcal{U}}(v) = v$ if $v \in \mathcal{U}$. Given two vectors $x$ and $y$, their stacking $[x^T \ y^T]^T$ will be denoted $(x, y)$. The $L_2$ norm of a signal $w(t)$ is defined as $\|w\|_2 := \sqrt{\int_0^\infty [w(t)]^2 \, dt}$, and $w \in L_2$ if $\|w\|_2 < \infty$. A system $\Sigma$ with input $(u, v)$ and output $(y, z)$ is said to have finite incremental ($L_2$ induced) gain $\gamma_{\Sigma} \in \mathbb{R}_{\geq 0}$ from $u$ to $y$ if for any initial condition $q_0$, and any pair of inputs $u_1(\cdot)$, $u_2(\cdot)$, it holds that

$$\|y(\cdot; q_0, u_1, v) - y(\cdot; q_0, u_2, v)\|_2 \leq \gamma_{\Sigma} \|u_1 - u_2\|_2,$$

where $y(t; q_0, u, v)$ is the output response at time $t$ to the initial condition $q_0$ and inputs $u(\cdot), \ v(\cdot)$. If $\Sigma$ is linear time invariant (LTI) with transfer function $W(s)$, its incremental
gain is equal to \( \|W(s)\|_\infty := \sup_{\omega \in \mathbb{R}} \sigma(W(j\omega)) \), where \( \sigma(\cdot) \) is the maximum singular value of the argument. The trivial system (whose output is identically null for any input) is indicated by \( \tilde{0} \), and has zero incremental gain.

In order to ease the comparison with existing results \([15]\), \([14]\), \([2]\), the first part of the paper deals with uncertain systems \( P_0 \) obtained by the connection of a nominal model \( P \) and a “perturbation” \( \Psi \), where \( P \) is given by:

\[
\begin{align*}
\dot{x} &= Ax + B\delta + y\Psi, \\
z &= C_1x + D_1\delta, \\
y &= C_2x + D_2\delta,
\end{align*}
\]

where \( y \) is the measured output, \( z \) is a controlled output, \( \delta = \text{sat}(u) \) is a saturated version of the control input \( u \) (\( u \) in the linear model) and \( y\Psi \) is the output of a perturbation \( \Psi \) belonging to a family \( S \) of incrementally stable systems,

\[
\begin{align*}
\dot{\psi} &= f_\Psi(\psi, x, \delta), \\
y\Psi &= h_\Psi(\psi, x, \delta),
\end{align*}
\]

(different elements of \( S \) can have different state spaces). Obviously, \( 0 \in S \) is assumed, so \( P_0 \) is the nominal model.

For \( \rho \in \mathbb{R}_{>0}, \) define the family \( S_\rho := \{ \Psi \in S : \gamma_{\Psi, \text{aw}, u_0} < \rho \} \), i.e. the subset of \( S \) containing only uncertainties with incremental gain less than \( \rho \) from \( u_0 = (x, u) \) to \( y\Psi \). A property (e.g., \( L_2 \) stability) possibly enjoyed by a system \( \Sigma_\Psi \) depending on a parameter \( \Psi \) in \( S \) is nominal if enjoyed by \( \Sigma_\Psi \) when \( \Psi = 0 \), it is robust in the small (with respect to \( S \)) if enjoyed by \( \Sigma_\Psi \) for all \( \Psi \in S_\rho \) for some \( \rho \in \mathbb{R}_{>0}, \) it is robust in the large (with respect to \( S \)) if enjoyed by \( \Sigma_\Psi \) for all \( \Psi \in S \). In other words, a property is robust in the small if it holds for a subset of \( S \) containing only \( \Psi \)’s having sufficiently small incremental gain.

In the anti-windup (aw) problem, a controller \( K_M \) designed for system (1) is given (\( r \) is a reference signal):

\[
\begin{align*}
\dot{x}_c &= A_cx_c + B_eu_c + E_cr, \\
y_c &= C_cx_c + D_1u_c + F_cr,
\end{align*}
\]

and the goal is to design an aw compensator \( K_{AW} \):

\[
\begin{align*}
\dot{x}_{aw} &= f_{aw}(x_{aw}, y, x_c), \\
v_1 &= h_{aw,1}(x_{aw}, y, x_c), \\
v_2 &= h_{aw,2}(x_{aw}, y, x_c),
\end{align*}
\]

which, suitably connected to \( P_\Psi \) and \( K_M \), ensures some nice properties for the overall closed loop system. Since it will be useful to use shorthand notations to refer to different interconnections of \( P_\Psi \), \( K_M \) and \( K_{AW} \), define the following closed loop systems (cls): (1), (2), (3) form the unsaturated cls \( \Sigma_U \) when \( u = y_c, u_c = y_c \), and the saturated cls \( \Sigma_S \) when \( u = \text{sat}(y_c), u_c = y_c \); (1), (2), (3), (4) form the unsaturated aw cls \( \Sigma_{U\text{AW}} \) when \( u = y_c + v_1, u_c = y + v_2 \), and the (saturated) aw cls \( \Sigma_{S\text{AW}} \) when \( u = \text{sat}(y_c + v_1), u_c = y + v_2 \). Different “hats” denote a signal related to a system (e.g., the state \( x \) of \( P \)) in a particular cls; \( \hat{\cdot} \) denotes the signal in \( \Sigma_U \) (e.g., \( \hat{x} \) for the state of \( P \) as a subsystem of \( \Sigma_U \)), \( \check{\cdot} \) denotes the signal in \( \Sigma_{U\text{AW}} \) (e.g., \( \check{x} \) for the state of \( P \) as a subsystem of \( \Sigma_{U\text{AW}} \)), and no hat denotes the signal in \( \Sigma_{S\text{AW}} \) (e.g., \( x \) for the state of \( P \) as a subsystem of \( \Sigma_{S\text{AW}} \)).

2. A Robust Anti-Windup Compensator

For the case of an open loop asymptotically stable plant, the global \( L_2 \) aw problem and its solution were defined in [15] as follows (the statements here have been slightly modified for notational coherence; [15] contains local results for the unstable case, too). In the definition, \( U \) denotes an arbitrary compact subset of the interior of the region where the saturation is linear.

**Definition 2.1:** The robust (in the small), \( L_2 \) aw problem for \( U \subset \mathbb{R}^p \) and \( S \) is to find an aw compensator such that

1) if \( x_{aw}(0) = 0 \) and \( \check{u}(\cdot) \equiv \text{sat}(\hat{u}(\cdot)) \) then \( z(\cdot) \equiv \check{z}(\cdot) \);
2) if \( \text{dist}_{\mathcal{M}}(\hat{u}(\cdot)) \in L_2 \) then \( (z - \check{z})(\cdot) \in L_2 \);

for all \( \Psi \in S \) with sufficiently small incremental gain.

Notice that, in order for the above problem to make sense, the following assumption was introduced in [15].

**Assumption 2.2:** The cls \( \Sigma_{U_0} \) is well-posed and internally stable \( \forall \Psi \in S, \) and \((C_1, A)\) is detectable.

Unfortunately, by means of Assumption 2.2 it is only possible to guarantee robustness in the small, and in fact for the large parameter variations considered in this paper for the TFA model and the given nominal compensator, the direct application of the extensions to the unstable case [14], [2] of the solution of the \( L_2 \) aw problem is not possible, due to a form of instability which can arise in any aw compensation satisfying item 1 in Definition 2.1 when large uncertainties are allowed [7] (see the discussion on \( F \circ \Psi \) after Definition 2.5). Since for the TFA model and the given nominal controller the problem in Definition 2.1 cannot be solved robustly in the large (at least without using any measure of the dynamic pressure), the requirements in Definition 2.1 must be relaxed in order to have a meaningful problem (whose solution gives a well-behaved control system \( \Sigma_{U\text{AW}} \) which is solvable robustly in the large; this can be done as in the weakened global \( L_2 \) aw problem introduced in [7], which is extended in this paper to provide an aw compensator for the unstable TFA model. To start with, the weakened aw problem is recalled in Definition 2.5. Two mild assumptions (recall that, with bounded inputs, global asymptotic controllability robust to arbitrary small errors in \( A \) requires that \( A \) is Hurwitz) are needed for the problem to make sense.

**Assumption 2.3:** The cls \( \Sigma_{U_0} \) is well-posed and internally stable \( \forall \Psi = 0 \).

**Assumption 2.4:** \( \exists \gamma_{\Psi, \text{aw}, u} : (P_\Psi, (P_0)_{u \in \mathbb{R}_{>0}}) \in \mathbb{R}_{>0} \) such that

\[
\begin{align*}
\| x(\cdot; (x_0, \psi_0), u_1) - x(\cdot; (x_0, \psi_0), u_2) \|_2 & \leq \gamma_{\Psi, \text{aw}, u}(P_\Psi) \| v_1 - v_2 \|_2; \quad \forall \Psi \in S \\
\| y_\Psi(\cdot; (x_0, \psi_0), u_1) - y_{\Psi}(\cdot; (x_0, \psi_0), u_2) \|_2 & \leq \gamma_{\Psi, \text{aw}, u}(P_\Psi) \| v_1 - v_2 \|_2; \quad \forall \Psi \in S.
\end{align*}
\]
Definition 2.5: The weakened global $\mathcal{L}_2$ aw problem for $\mathcal{U}$ with domain of robustness $\mathcal{S}$ is to find an aw compensator such that:

1) for $\Psi = 0$, $\exists x_0:\text{ if } x_{aw}(0) = x_0^{aw}$ and $\bar{u}(\cdot) \equiv \text{sat}(\bar{u}(\cdot))$, then $z(\cdot) \equiv \bar{z}(\cdot)$;
2) $\Sigma_{U, \text{aw}}$ is well-posed and internally stable $\forall \Psi \in \mathcal{S}$;
3) if $\text{dist}_{U}(\bar{u}(\cdot)) \in \mathcal{L}_2$ then $(z - \bar{z})(\cdot) \in \mathcal{L}_2$. □

In order to understand why the problem in Definition 2.5 is a “weak” form of the problem in Definition 2.1, it must be noticed that the key difference between the two problems is a trade off between aw performance and robustness. In fact, item 1 of Definition 2.1 requires the small signal response of $\Sigma_{S, \text{aw}}$ to match the response of $\Sigma_U$ for all possible perturbations, whereas item 1 of Definition 2.5 is only the corresponding nominal ($\Psi = 0$) requirement. The robust/nominal dichotomy is reflected in the fact that Assumption 2.3 is only a nominal version of Assumption 2.2. Both item 3 of Definition 2.5 and item 2 of Definition 2.1 assess the effectiveness of the aw response for large signals by limiting the $\mathcal{L}_2$ difference between the responses of $\Sigma_{S, \text{aw}}$ and another system which is well-behaved robustly with respect to $\mathcal{S}$; however, the comparison system in Definition 2.5 is $\Sigma_{U, \text{aw}}$ and not $\Sigma_U$ (which need not even be stable for all $\Psi \in \mathcal{S}$); instead, robust well-posedness and stability of the comparison system $\Sigma_{U, \text{aw}}$ is required in item 2 of Definition 2.5) as in Definition 2.1. Since robust stability of $\Sigma_U$ is not required, the design of $K_M$ can be completely focused on performance, while robustness will be in charge of the aw compensator, as will be explained now.

The proofs of the theorems that solve the problems above [7], [15] exploit a coordinate transformation by which $\Sigma_{S, \text{aw}}$ is rewritten as an equivalent system (depicted in Fig. 3) in which the unmeasured signal $y_\bar{u}$ appears “filtered” by an LTI n--input, n--output system $F$ with state $x_F \in \mathbb{R}^{n_F}$, where $F = I$ in the case of $\mathcal{L}_2$ aw, whereas $F(s)$ is an arbitrary rational proper transfer matrix in the case of weakened aw. The robustness properties of the proposed solutions are shown exploiting the fact that for $F \circ \Psi = 0$, the equivalent system has suitable stability properties, that are preserved (using a small gain argument) either if $F(s) = I$, Assumption 2.2 holds and $\Psi$ is sufficiently small (as in $\mathcal{L}_2$ aw), or if $\Psi \in \mathcal{S}$, Assumption 2.4 holds and $K$, $F(s)$ are “suitably small” (as in weakened aw). In general, the small gain condition allows for enough freedom to guarantee additional properties for $\Sigma_{S, \text{aw}}$ by suitable choice of $F(s)$ [7]; in particular, though $F = 0$ would guarantee $F \circ \Psi = 0$ (easing the achievement of robust stability), when specific constant [or sinusoidal $r(t) = \sin(\omega t + \phi)$] signals must be tracked, it is desirable to have $F(0) = I$ [$F(j\omega) = I$]. For the TAF model, the instabilities witnessed for large $\Psi \in \mathcal{S}$ when $\mathcal{L}_2$ aw is employed are due to the fact that $F \circ \Psi = \Psi$ makes $\Sigma_{U, \text{aw}}$ unstable for such $\Psi$’s (though, by Assumption 2.2, $\Sigma_U$ is asymptotically stable $\forall \Psi \in \mathcal{S}$); however, since for the TAF manual control the “steady state” (cruise) value of the reference (the desired pitch rate) can be considered zero after some “transients” (maneuvers), the choice $F = 0$ is feasible from the tracking point of view, and can be used to solve a weakened aw problem. Moreover, though in [15], [7] the scheme in Fig. 3 is just an analysis tool used in the proofs (direct implementation of the dashed arrows is impossible since $y_\bar{u}$ cannot be measured!), when $F = 0$ the scheme can be used for implementation too. Though in a somewhat different framework, similar “model following” control systems have been proposed for a long time (see e.g. [12], [16]; see also [10] for an application to flight control).

There are two main issues to be dealt with in the required extension: first, the aw compensator must guarantee that the state of the controlled system never leaves the safe region $\mathcal{V}$; second, Assumption 2.4 (which was quite mild in the global case since asymptotic stability of the perturbed plant was “almost necessary”) must be replaced by some other suitable assumption, since even the nominal system is now unstable and then its $\mathcal{L}_2$ gain is not defined. Both issues are taken into account on the basis of the following assumption, basically requiring the knowledge of a robust stabilizer with suitable properties. In the statement, $(x_M(\cdot), u_M(\cdot))$ is any couple of measurable time functions with $x_M(t) \in \mathbb{R}^n$, $u_M(t) \in \mathbb{R}^p$. A convex safe region $\mathcal{V} \subset \mathbb{R}^n$ is given, and it is desired to have $x(t) \in \mathcal{V}$, $\forall t \geq 0$.

Assumption 2.6: There exists a function $k(\cdot, \cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^p$, and two compact convex sets $\mathcal{X} \subset \mathbb{R}^n$, $\mathcal{X}_0 \subset \mathbb{R}^n$ with $0 \in \mathcal{X} \subset \mathcal{V}$, $0 \in \mathcal{X}_0 \subset \text{int} \mathcal{X}$, such that:

1. $k(x, x, u_M) = u_M$, $\forall (x, u_M) \in \mathcal{X}_0 \times \mathcal{U}$;
2. if $x(0) \in \mathcal{X}$ then $t(x(0), k(x, x_M(t), u_M(t))) \in \mathcal{V}$,
   $\forall t \geq 0$, $\forall \Psi \in \mathcal{S}$, $\forall (x_M(\cdot), u_M(\cdot))$;
3. if $\lim_{t \to +\infty} (x_M(t), u_M(t)) = 0$ and $x(0) \in \mathcal{X}$ then
   $\lim_{t \to +\infty} t(x(0), k(x(t), x_M(t), u_M(t))) = 0$, $\forall \Psi \in \mathcal{S}$.

Though the design of $k(\cdot, \cdot, \cdot)$ may look a quite hard task, in many practical cases the easy approach used for the TAF model in Section 3 can be effectively used.

Definition 2.7: The weakened aw problem for $\mathcal{U}$ with domain of robustness $\mathcal{S}$, domain of attraction $\mathcal{X}$, domain of nominal behaviour $\mathcal{X}_0 \times \mathcal{U}$, state constraint $\mathcal{V}$ and asymptotically null references, is to find an aw compensator such that, for all $x(0) \in \mathcal{X}$,

1) for $\Psi = 0$, $\exists x_0^{aw}$ if $x_{aw}(0) = x_0^{aw}$, $\Gamma_{\mathcal{X}_0}(\bar{x}(\cdot)) = \bar{x}(\cdot)$,
   $\Gamma_{\mathcal{X}_0}(\bar{z}(\cdot)) = \bar{z}(\cdot)$, then $\bar{z}(\cdot) \equiv \bar{z}(\cdot)$;
2) $\Gamma_{\mathcal{X}}(x(\cdot)) = x(\cdot)$, $\forall \Psi \in \mathcal{S}$;
3) if $\lim_{t \to +\infty} x(t) = 0$ then $\lim_{t \to +\infty} x(t) = 0$, $\forall \Psi \in \mathcal{S}$.
In Definition 2.7, item 1 is the classical requirement of matching the response induced by the nominal controller, restricted to the nominal parameter ($\Psi = 0$) case, as in weakened aw, and to the region $X_0 \times U$, since following the nominal response outside such region could lead to the state outside $V$; item 2 guarantees that the safe region $V$ is never left; finally, item 3 is a weakened replacement of item 3 (and item 2, in some sense) of Definition 2.5: in fact, assuming $\Sigma_{U_AW}$ well-posed and internally stable $\forall \Psi \in S$, and asymptotically null references, bears that $\dot{z}$ is asymptotically zero, and then item 3 of Definition 2.5 can be replaced by the requirement that $x$ converges to zero (which, taking into account that $x$ is a bounded signal, is something less than asking $x \in L_2$) and there is no need to explicitly require well-posedness and asymptotic stability of $\Sigma_{U_AW}$, and then item 2 of Definition 2.5 is dropped. The following theorem provides a solution to the problem in Definition 2.7.

**Theorem 2.8:** Under Assumption 2.3 and Assumption 2.6, the following aw compensation having state $x_{aw} \in \mathbb{R}^n$:

\[
\begin{align*}
\dot{x}_{aw} &= Ax_{aw} + B y_c, \\
v_1 &= -y_c + k(x, x_{aw}, y_c), \\
v_2 &= -y_c + C_2 x_{aw} + D_2 y_c,
\end{align*}
\]

with $x_{aw}(0) = x(0)$ solves the problem in Definition 2.7. □

**Proof:** With the above described aw compensation, the overall closed loop is exactly the system in Fig. 3, where $(x_M, u_M)$ is $(x_{aw}, y_c)$ and $K$ implements the control law $k(x, x_{aw}, u_M)$. Items 1-3 of Definition 2.7 are then straightforward consequences of Assumption 2.6.

3. Results

In order to apply Theorem 2.8 to the robust aw synthesis for the reduced short-period longitudinal model of the TAFA, in this section first the linearized model used in [2] is briefly recalled and the family $S$ of uncertainties is characterized; then, a procedure that can be used to determine $\mathcal{V}$, $\mathcal{X}$, $X_0$, $k(\cdot, \cdot, \cdot)$ is outlined, and the resulting sets and functions are explicitly given; finally, numerical simulations are performed and commented upon.

The linearized model of interest is described, for dynamic pressure $Q = 0.5pV^2$ in the range 100-1200 psf, by [2]:

\[
\begin{align*}
\dot{x} &= (A + \bar{Q} \Delta_x)x + (B + \bar{Q} \Delta_u) \text{sat}_M(u), \\
y &= z = x,
\end{align*}
\]

where $x = (\alpha, q)$ is the state ($\alpha$ is the angle of attack, $q$ is the body axis pitch rate) coinciding with both the measured and the performance output, $\bar{Q}$ is the difference between the actual $Q \in (100, 1200)$ and the nominal $Q_0$ value of the dynamic pressure, the actuator saturation limit is $M = 0.35$ rad, the nominal $(\bar{Q} = 0)$ dynamics at trim condition $Q_0 = 450$ psf is

\[
A = \begin{bmatrix} Z_0^0 & Z_q^0 \\ M_0^0 & M_q^0 \end{bmatrix} = \begin{bmatrix} -0.9 & 1.0 \\ 5.9375 & -2.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ M_0^0 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \end{bmatrix},
\]

the family $S$ of uncertainties $\Psi$ is $y_{\Psi} = \bar{Q} \Delta u_{\Psi}$, with $u_{\Psi} = (x, \text{sat}_M(u))$, $\bar{Q} \in (-350, 750)$ and

\[
\Delta = [\Delta_x \quad \Delta_u] = \begin{bmatrix} \frac{1}{160} & 0 & 0 \\ \frac{1}{300} & \frac{4}{255} \end{bmatrix}.
\]

As shown in Fig. 1, the plant has two real eigenvalues for $Q > 128$, one of which becomes unstable for $Q > 165.3$. For fixed $Q > 165.3$, the system can be diagonalized

\[
\begin{bmatrix} \dot{x}_s \\ \dot{u}_s \end{bmatrix} = \lambda_s \begin{bmatrix} x_s \\ u_s \end{bmatrix} + \begin{bmatrix} b_s \\ b_u \end{bmatrix} \text{sat}_M(u),
\]

so that the null controllability region in the $(x_s, u_s)$ variables is $\mathcal{V}_Q = \{(x_s, u_s) \in \mathbb{R}^2 : |x_s| < M|b_u|/|\lambda_u|\}$; in the $(\alpha, q)$ variables, such a stripe will appear transformed by the diagonalizing change of coordinates (see the bold lines in Fig. 2, corresponding to $Q = Q_0$). When $Q$ varies, both $\lambda_u$ and its eigenvector change, determining the convex safe region $\mathcal{V} = \bigcap_{Q \in (100, 1200)} \mathcal{V}_Q$ (the central white area in Fig. 2) such that if $x(t) \not\in \mathcal{V}$ for some $t \geq 0$, then the state is outside the null controllability region for at least a value of $Q$, and then is unrecoverable (at least as long as $Q$ is constant at such value): this motivates the need to have $x(t) \in \mathcal{V}$, $\forall t \geq 0$. In order to construct our aw compensator, 1) we look for a gain matrix $K$ such that $A + \bar{Q} \Delta_x + (B + \bar{Q} \Delta_u)K$ is Hurwitz for all admissible values of $Q$; 2) we approximate $\mathcal{V}$ by a set $\mathcal{V}_0 \subset \mathcal{V}$ (the dashed parallelogram in Fig. 2); 3) we find a set $\mathcal{X} \subset \mathcal{V}_0$ (the dash-dotted parallelogram in Fig. 2) such that for $x = (A + \bar{Q} \Delta_x)x + (B + \bar{Q} \Delta_u) \text{sat}_M(Kx)$ all trajectories starting in $\mathcal{X}$ converge to the origin without exiting $\mathcal{V}_0$, for all admissible values of $Q$; 4) we blend the above ingredients to determine $k(\cdot, \cdot, \cdot)$.

Using as a first guess the value of $K = [-1.03 \quad -0.36]$ corresponding to $u = -(\lambda_u + \epsilon)x_u$ with $\epsilon = 0.52$ for the value of $Q = 412$ psf, minimizing $M|b_u|/|\lambda_u|$ for $Q \in (165, 1200)$ [the region where there is an unstable eigenvalue], it is easy to check that such a $K$ is a suitable choice for step (1) above. As for step (2), a good internal approximation $\mathcal{V}_0$ of $\mathcal{V}$ can be trivially found. As for step (3), a simple Matlab routine based on ode integration has been used to determine $\mathcal{X}$ as shown in Fig. 2. We remark that, though powerful tools based on set invariance and output admissible sets [4], [9] can be used to derive the largest possible sets $\mathcal{V}$ and $\mathcal{X}$, their use in the present paper has been avoided in order to show that (at least for the considered TAFA model) a much simpler yet effective approach can be used instead.

Letting $\gamma \in (0, 1)$ (in the simulation, we chose $\gamma = 0.95$), $x_0 = \{x : \gamma^{-1}x \in \mathcal{X}\}$, a function $k(x, x_M, u_M)$ satisfying Assumption 2.6 can be easily given in the form:

\[
k(k(x, x_M, u_M)) = K(x - \beta(x)\Gamma_{x_0}(x_M)) + \beta(x)\Gamma_{U_0}(u_M),
\]

where $\beta(x)$ is any continuous function such that $\beta(x) = 1$ if $x /in x_0$, and $\beta(x) = 0$ if $x \not\in \mathcal{X}$, and the projections are $\Gamma_{x_0}(x_M) = \max\{1, F_1 x_M, F_2 x_M, F_3 x_M, F_4 x_M\}$ and $F_i$, $i = 1, \ldots, 4$ are such that $x_M \in x_0$ Iff $F_i x_M \leq 1$,
for i = 1, . . . , 4 [5] and \( \Gamma_{U_i}(u_M) = \text{sat}_M(u_M) \) (so that \( U_i = U \); the function \( \Gamma_{U_i}(\cdot) \), tuned by trial and error by the choice of a convex compact \( U_i \supseteq U \), can be used to reduce some transients induced by excessive values of \( u_M \)).

The a priori given, nominal controller designed in [2] in order to meet a prototypical military specification \( U \) choice of a convex compact \( \Sigma \) is present also for the nominal plant (when saturation of the elevator is encountered. This instability inherent capability of angle of attack limiting is achieved, that to large variations of the trim condition. Though the current overall aw closed loop system is required to be robust when only magnitude saturation is taken into account, but the additional issue of rate saturation is also a subject of current investigation, as well as the feasibility of our design for higher order models, containing either a more complete dynamical description (e.g. a complete longitudinal model) and/or additional dynamical uncertainties (neglected actuator dynamics or flexural modes).

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### References


