On output feedback robustified anti-windup compensators

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Abstract

Conditions are given under which it is possible to implement a compensator solving a weakened anti-windup problem without full state measurements. The effectiveness of unknown input observers is also shown for the cases when the above conditions are not satisfied.

1. Introduction

The performance (or even stability) loss due to the presence of saturation in control systems designed without taking into account the limits on the control magnitude is widely known as “windup phenomenon”; an anti-windup (aw) compensator is an additional controller that, suitably connected to the saturating control system, (1) does not influence the response of such system as long as the input generated by the nominal controller is smaller than the saturation limit, but (2) can modify the input otherwise in order to limit the performance loss due to saturation. Due to the ubiquitous presence of saturation, a lot of research has been devoted to this problem, ranging from heuristic techniques for specific cases when the above conditions are not satisfied.

Recently, it has been pointed out in [2, 3] that the very definition of aw compensation in terms of the two properties above can imply a lack of robustness for the overall anti-windup closed loop system; in particular, even if the linear closed loop is robustly stable for a whole family of perturbations, it may happen that any aw compensator (regardless the specific technique adopted for its synthesis) for such control system satisfying properties (1) and (2) will only guarantee stability for a possibly much smaller family of perturbations. Such a fact motivated the introduction of a weakened aw problem (a modification of the $L_2$-aw [7] problem) and its solution in [2, 3]. An unattractive feature of the aw compensator proposed in [2, 3] is the need for full state measurements, a characteristic in sharp contrast with other techniques available in the literature (e.g., the compensator in [7] does not require any measurement from the plant).

The main purpose of the present paper is to investigate whether the need for full state measurements in the solution of the weakened aw problem can be relaxed, possibly by introducing an estimator of the state of the plant. In the process of answering the above question, a wider perspective is taken by showing the underlying structure enabling the solution of the weakened aw problem, and some of its implications. The structure of the paper is as follows: Sec. 2 gives a brief account of what the weakened aw problem is, and points out its peculiarities by comparing it with the $L_2$-aw problem; Sec. 3 discusses the output feedback implementation issues; Sec. 4 shows a simple example substantiating the performed analysis; conclusions and directions for future work are given in Sec. 5.

Notation

For a given convex set $U \subset \mathbb{R}^p$ and a vector $u \in \mathbb{R}^p$, let $\text{dist}_U(u) := \inf_{w \in U} |u - w|$, where $| \cdot |$ represents the Euclidean norm. Given two vectors $x$ and $y$, their stacking $[x' \ y']'$ will be denoted $(x, y)$. The $L_2$ norm of a signal $w(\cdot)$ is defined as $\|w\|_2 := \sqrt{\int_0^\infty |w(t)|^2 \, dt}$, and $w \in L_2$ if $\|w\|_2 < \infty$. A system $\Sigma$ with input $(u, v)$ and output $(y, z)$ is said to have finite incremental ($L_2$ induced) gain $\gamma_{y, u}^\Sigma \in \mathbb{R}_{\geq 0}$ from $u$ to $y$ if for any initial condition $\sigma_0$, and any pair of inputs $u_1(\cdot), u_2(\cdot)$, it holds that

$$\|y(\cdot; \sigma_0, u_1, v) - y(\cdot; \sigma_0, u_2, v)\|_2 \leq \gamma_{y, u}^\Sigma \|u_1 - u_2\|_2,$$

where $y(t; \sigma_0, u, v)$ is the output response at time $t$ to the initial condition $\sigma_0$ and inputs $u(\cdot), v(\cdot)$. If $\Sigma$ is linear time invariant (LTI) with transfer function $W(s)$, its incremental gain is equal to $\|W(s)\|_\infty := \sup_{\omega \in \mathbb{R}} |\mathcal{P}(W(j\omega))|$, where $\mathcal{P}(\cdot)$ is the maximum singular value of the argument. The trivial system (whose output is identically null for any input) is denoted by $0$, and has zero incremental gain.

The paper deals with uncertain systems $P_\Psi$ obtained by the connection of a nominal model $P$ and a “pertur-
bation” $\Psi$, where $P$ is given by:

$$
\begin{align}
\dot{x} &= Ax + B_1d + B_2u + y\varphi, \quad (1.1a) \\
z &= C_1x + D_{11}d + D_{12}u, \quad (1.1b) \\
y &= C_2x + D_{21}d + D_{22}u, \quad (1.1c)
\end{align}
$$

where $y$ is the measured output, $z$ is a controlled output, $u$ is a control input, $d$ is an exogenous disturbance, and $y\varphi$ is the output of a perturbation $\Psi$ belonging to a family $S$ of incrementally stable systems, where any $\Psi \in S$ can be described by

$$
\begin{align}
\dot{\psi} &= f_{\Psi}(\psi, x, u, d), \quad (1.2a) \\
y_{\Psi} &= h_{\Psi}(\psi, x, u, d), \quad (1.2b)
\end{align}
$$

(different elements of $S$ can have different state spaces).

Obviously, $0 \in S$ is assumed, so $P_0$ is the nominal model.

For $\rho \in \mathbb{R}_{>0}$, define the family $S_{\rho} := \{\Psi \in S : \gamma(y_{\Psi}, u\varphi) < \rho\}$, i.e., the subset of $S$ containing only uncertainties with integral gain less than $\rho$ from $u\varphi = (x, u)$ to $y\varphi$. A property (e.g., $L_2$ stability) possibly enjoyed by a system $\Psi$ depending on a parameter $\Psi \in S$ is nominal if enjoyed by $\Sigma_{\Psi}$ when $\Psi = 0$, it is robust in the small (with respect to $\Psi$) if enjoyed by $\Sigma_{\Psi}$ for all $\Psi \in S_{\rho}$ for some $\rho \in \mathbb{R}_{>0}$, it is robust in the large (with respect to $\Psi$) if enjoyed by $\Sigma_{\Psi}$ for all $\Psi \in S$. In other words, a property is robust in the small if it holds for a subset of $S$ containing only $\Psi$’s having sufficiently small incremental gain.

In the aw problem, a controller $K_M$ designed for system $\Sigma_U$ is supposed to be given:

$$
\begin{align}
\dot{x}_c &= A_cx_c + B_cu_c + E_c\phi, \quad (1.3a) \\
y_c &= C_cx_c + D_cu_c + E_c\phi, \quad (1.3b)
\end{align}
$$

and the goal of aw synthesis is to design an additional aw compensator $K_{AW}$ of the form

$$
\begin{align}
\dot{x}_{aw} &= f_{aw}(x_{aw}, y_c), \quad (1.4a) \\
v_1 &= h_{aw,1}(x_{aw}, y_c), \quad (1.4b) \\
v_2 &= h_{aw,2}(x_{aw}, y_c), \quad (1.4c)
\end{align}
$$

which, suitably connected to $P_0$ and $K_M$, ensures some nice properties for the overall closed loop system. Since it will be useful to have shorthand notations to refer to different interconnections of $P_0$, $K_M$ and $K_{AW}$, define the following closed loop systems (cls): the unsaturated cls $\Sigma_U$, given by (1.1), (1.2), (1.3) with the interconnection $u = y_c$, $u_c = y$; the saturated cls $\Sigma_S$, given by (1.1), (1.2), (1.3) with the interconnection $u = \text{sat}(y_c)$, $u_c = y$; the unsaturated aw cls $\Sigma_{U Aw}$, given by (1.1), (1.2), (1.3), (1.4) with the interconnection $u = y_c + v_1$, $u_c = y + v_2$; the (saturated) anti-windup cls $\Sigma_{SAW}$, given by (1.1), (1.2), (1.3), (1.4) with the interconnection $u = \text{sat}(y_c + v_1)$, $u_c = y + v_2$.

Different “hats” denote a signal related to a system (e.g., the state $x$ of $P$) in a particular cls: $\hat{\phi}$ denotes the signal in $\Sigma_U$ (e.g., $\dot{x}$ for the state of $P$ as a subsystem of $\Sigma_U$), $\bar{\phi}$ denotes the signal in $\Sigma_{U Aw}$ (e.g., $\dot{x}$ for the state of $P$ as a subsystem of $\Sigma_{U Aw}$), and no hat denotes the signal in $\Sigma_{SAW}$ (e.g., $x$ for the state of $P$ as a subsystem of $\Sigma_{SAW}$).

As in [7], a function $\sigma : \mathbb{R}^p \to \mathbb{R}^p$ is considered a “saturation” function if $\exists U \subset \mathbb{R}^p$, $U$ compact and convex, and $\exists L, b \in \mathbb{R}_{>0}$ such that $|\sigma(u + \varphi) - (u + \varphi)| \leq LU|\sigma(u + \varphi) - \varphi|$ and $|\sigma(u + w) - \sigma(u)| \leq \min\{L|w|, b\}$, $\forall u, w \in \mathbb{R}^p$, $\forall \varphi \in U$.

Clearly, the previous class contains the “standard” decentralized saturation function for which $y = \text{sat}(u)$ means $y_i = \text{sign}(u_i)\min\{|u_i|, u_{i,MAX}\}$, if $U$ is chosen as any compact subset of the interior of the region where the saturation is linear.

2. The weakened aw problem

The weakened global $L_2$ aw problem defined in [2] is recalled in Definition 2.3. Two mild assumptions (recall that, with bounded inputs, global asymptotic controllability robust to arbitrary small errors in $A$ requires that $A$ is Hurwitz) are needed for the problem to make sense.

Assumption 2.1. The cls $\Sigma_U$ is well-posed and internally stable for $\Psi = 0$. \hfill $\Box$

Assumption 2.2. $\exists \gamma_{\Sigma_{U Aw}}$ such that

$$
\|x(\cdot; (x_0, \psi_0), u_1, d) - x(\cdot; (x_0, \psi_0), u_2, d)\|_{2} \leq \gamma_{\Sigma_{U Aw}} \|u_1 - u_2\|_{2}, \quad \forall \Psi \in S
$$

$$
\|y_{\Psi}(\cdot; (x_0, \psi_0), u_1, d) - y_{\Psi}(\cdot; (x_0, \psi_0), u_2, d)\|_{2} \leq \gamma_{\Sigma_{U Aw}} \|u_1 - u_2\|_{2}, \quad \forall \Psi \in S. \quad \Box
$$

Definition 2.3. The weakened global $L_2$ aw problem for $U$ domain of robustness $S$ is to find an aw compensator such that:

1. for $\Psi = 0$ and $d = 0$, $\exists x_{aw}^{0}$; if $x_{aw}(0) = x_{aw}^{0}$ and $\bar{u}(\cdot) \equiv \text{sat}(\bar{u}(\cdot))$, then $z(\cdot) \equiv \bar{z}(\cdot)$;

2. $\Sigma_{U Aw}$ is well-posed and internally stable $\forall \Psi \in S$;

3. if $\text{dist}(\bar{u}(\cdot)) \in L_2$ then $(\cdot, \bar{z}(\cdot)) \in L_2$. \hfill $\Box$

In the following theorem, $F = (A_F, B_F, C_F, D_F)$ is an LTI $n$–input, $n$–output system with state $x_F \in \mathbb{R}^{n_F}$ and transfer matrix $F(s)$.

Theorem 2.4. Under Assumption 2.1 and Assumption 2.2, $\exists \gamma_F^{*} \in \mathbb{R}_{>0}$; $\gamma_F^{*} : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that $\forall K \in \mathbb{R}^{n \times n}$ with $\|K\| < \gamma_F^{*}$, $VF(s)$ satisfying...
\( \bar{\sigma}(F(j\omega)) < \gamma_F^r(\omega), \forall \omega \in \mathbb{R}_{>0} \), the following aw compensation having state \( x_{aw} \in \mathbb{R}^{n+m} \):

\[
\dot{x}_{aw} = \begin{bmatrix} A_F & 0 \\ -C_F & A \end{bmatrix} x_{aw} + \begin{bmatrix} A_F B_F - B_F A \\ D_F A - A D_F - C_F B_F \end{bmatrix} x \\
+ \begin{bmatrix} -B_F B_2 \\ D_F B_2 \end{bmatrix} \text{sat}(y_c + v_1) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} y_c, \tag{2.1a}
\]

\[
\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & K \\ 0 & -C_2 \end{bmatrix} x_{aw} + \begin{bmatrix} K(I - D_F) \\ -C_2(I - D_F) \end{bmatrix} x \\
- \begin{bmatrix} 0 \\ D_2 \end{bmatrix} \text{sat}(y_c + v_1) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} y_c. \tag{2.1b}
\]

solves the problem in Definition 2.3.

To explain why the above problem is a “weak” form of aw problem, the definition of the global \( \mathcal{L}_2 \) anti-windup problem and its solution [7] are reported next (with slightly modified statements for notational coherence).

**Assumption 2.5.** The cls \( \Sigma_U \) is well-posed and internally stable \( \forall \Psi \in \mathcal{S} \), and \((C_1, A)\) is detectable. \( \square \)

**Definition 2.6.** The robust (in the small), global \( \mathcal{L}_2 \) aw problem for \( U \subset \mathbb{R}^p \) and \( \mathcal{S} \) is to find an aw compensator such that

1. if \( x_{aw}(0) = 0 \) and \( \bar{u}(\cdot) \equiv \text{sat}(\bar{u}(\cdot)) \) then \( z(\cdot) \equiv \bar{z}(\cdot); \)
2. if \( \text{dist}_U(\bar{u}(\cdot)) \in \mathcal{L}_2 \) then \( \bar{z}(\cdot) \in \mathcal{L}_2; \)

for all \( \Psi \in \mathcal{S} \) with sufficiently small incremental gain. \( \square \)

**Theorem 2.7.** Under Assumption 2.5, the problem in Definition 2.6 is solvable iff \( A \) is Hurwitz, and a possible aw compensation is:

\[
\dot{x}_{aw} = Ax_{aw} + B_2 \text{sat}(y_c + v_1) - y_c; \tag{2.2a}
\]

\[
v_1 = -B'_P x_{aw}; \tag{2.2b}
\]

\[
v_2 = -C_2 x_{aw} - D_{22} \text{sat}(y_c + v_1) - y_c. \tag{2.2c}
\]

with \( P \in \mathbb{R}^{n \times n} \), \( P = P' > 0 \); \( A' P + PA < 0 \). \( \square \)

The key difference between the two problems is a trade off between aw performance and robustness. In particular, item 1 of Definition 2.6 requires that the small signal response of the aw cls \( \Sigma_{SAW} \) exactly matches the response of the linear cls \( \Sigma_U \) for all possible perturbations \( \Psi \in \mathcal{S} \): such a matching property can be seen as a robust in the large performance requirement; on the other hand, item 1 of Definition 2.3 is only the corresponding nominal (for \( \Psi = 0 \), \( d \equiv 0 \)) performance requirement. The robust/nominal dichotomy is reflected in the fact that Assumption 2.1 is only a nominal (for \( \Psi = 0 \)) version Assumption 2.5. Both item 3 of Definition 2.3 and item 2 of Definition 2.6 assess the effectiveness of the aw response for large signals by limiting the \( \mathcal{L}_2 \) difference between the responses of \( \Sigma_{SAW} \) and another system which is well-behaved robustly with respect to \( \mathcal{S} \); however, contrary to what is done in Definition 2.6, the comparison system in Definition 2.3 is \( \Sigma_{UAW} \) and not \( \Sigma_U \) (which need not even be stable for all \( \Psi \in \mathcal{S} \); instead, robust well-posedness and stability of the comparison system \( \Sigma_{UAW} \) is required in item 2 of Definition 2.3). A positive consequence of not requiring robust stability of \( \Sigma_U \) is to decouple the problem of guaranteeing robustness from the design of controller \( K_M \) in (1.3), which can then be focused on performance. Finally, in Definition 2.3 the response from \( d \) to \( z \) is allowed to be modified if needed in order to be able to robustify the aw cls with respect to the whole class of uncertainties. Such a trade off is necessary since both \( d \) and \( y_P \) are not measured (so that their contributions in (1.1a) are in general not distinguishable), and then any “filtering” action performed on \( y_P \) in order to ensure robustness must also affect \( d \), thus modifying the response from \( d \) to \( z \) even for \( \Psi = 0 \).

Both theorems above [2, 7] are proved by using a coordinate transformation by which \( \Sigma_{SAW} \) is rewritten as an equivalent system in which the unmeasured signal \( y_P \) appears “filtered” by \( F = I \) (Theorem 2.7) and exploits the fact that for \( F \circ \Psi = 0 \), the equivalent system has suitable stability properties, so that by a small gain argument stability is preserved either by having \( F(s) = I \) and sufficiently small perturbations \( \Psi \) (as in Theorem 2.7), or by having \( \Psi \in \mathcal{S} \), Assumption 2.2 and a “suitably small” \( F(s) \) (as in Theorem 2.4). In general, the small gain condition allows for enough freedom to guarantee additional properties for \( \Sigma_{SAW} \) by choosing \( F(s) \) [2]; however, the price paid for \( F(s) \neq I \) in Theorem 2.4 is the need for full state measurements.

**Remark 2.8.** A straightforward (though not optimal) approach to the design of \( K \) and \( F \) as in Theorem 2.4 can be given as follows. Let \( T \) be the system with input \( u_T \), output \( y_T \) described by

\[
\begin{align*}
T : & \quad \dot{x}_M = A x_M + B_2 y_c + u_T, \\
& \quad \dot{x}_c = A_c x_c + B_3 y_c, \\
& \quad y_T = y_c - K_M y_c.
\end{align*}
\]

with \( y_c = C_c x_c + D_c y_M \) and \( y_M = C_2 x_M + D_21 d + D_{22} y_c \). Denote by \( T(s), F(s) \) the transfer matrices of \( T \) and \( F \). First, letting \( \gamma^*_K = (L^{(\psi_0)})^{-1}, K = -\rho B', \text{ fix } \rho \in \mathbb{R}_{>0} \) such that \( \|K\| < \gamma^*_K \); then, letting \( \bar{\gamma} = \frac{L^{(\psi_0)}}{1 - L^{(\psi_0)}}, \) and choose a stable \( F(s) \) such that \( \bar{\gamma} \|TF\|_{\infty} \leq 1 \). Clearly, \( F \) can be easily chosen as in classical loop-shaping / \( H_{\infty} \) design. Moreover, if all \( \Psi \in \mathcal{S} \) are LTl and a bound \( \psi(\omega) \) such that \( \bar{\sigma}(\Psi(j\omega)) \leq \psi(\omega), \forall \omega \in \mathbb{R} \) and \( \forall \Psi \in \mathcal{S} \) is known, the condition on \( \bar{\gamma} \|TF\|_{\infty} < 1 \) can be relaxed to \( \bar{\gamma} \|\psi(\omega)\|TF(j\omega)\| < 1, \forall \omega \in \mathbb{R} \). Such a weaker condition can be useful, for example, to preserve some asymptotic disturbance rejection behaviour induced by
between certain systems $\Sigma_1$ and $\Sigma_2$ (described below), with $\Sigma_1$ containing a direct filtering action on an unmeasured signal $d$, and $\Sigma_2$ similar to $\Sigma_{SAW}$. Three issues are considered: 1) is it possible to find an implementable (i.e., not measuring $d$) system $\Sigma_2$ equivalent to $\Sigma_1$? 2) is it possible to do so without full state measurements? and 3) if full state measurements are needed, is it possible to replace them with an asymptotic estimate of the state, at least in the solution of the weakened aw problem?

For compactness, in this section (1.1) is replaced by

$$
\begin{align*}
\dot{x} &= Ax + B_1d + B_2u, \\
y &= C_2x + D_{21}d,
\end{align*}
$$

(3.1a)

where $d$ is the unmeasured signal to be filtered, the performance output is dropped being irrelevant for the discussion to follow, and the term $D_{22}u$ can obviously be dropped without loss of generality. The extension to different patterns by which exogenous signals and/or perturbation outputs influence $\dot{x}$, $z$ and $y$ is straightforward. The basic problem to be addressed is the equivalence between the following clss $\Sigma_1$

$$
\begin{align*}
P_M: \quad \begin{cases}
\dot{x}_M = A_{1}x_M + B_1y_F + B_2y_c, \\
y_M = C_2x_M + D_{21}y_F + D_{22}y_c,
\end{cases}
\end{align*}
$$

(3.1b)

$$
\begin{align*}
K_M: \quad \begin{cases}
\dot{x}_c = A_c x_c + B_c y_M + E_c r, \\
y_c = C_c x_c + D_c y_M + F_c r,
\end{cases}
\end{align*}
$$

$$
\begin{align*}
F: \quad \begin{cases}
\dot{x}_F = A_F x_F + B_F y_d, \\
y_F = C_F x_F + D_F y_d,
\end{cases}
\end{align*}
$$

(3.1c)

$$
\begin{align*}
P: \quad \begin{cases}
\dot{x} = Ax + B_1d + B_2u, \\
y = C_2x + D_{21}d + D_{22}u, \\
u = \text{sat}(y_c + K(x - x_M)),
\end{cases}
\end{align*}
$$

(3.1d)

and the following clss $\Sigma_2$

$$
\begin{align*}
\dot{x_c} &= A_c x_c + B_c (y + v_2) + E_c r, \\
y_c &= C_c x_c + D_c (y + v_2) + F_c r,
\end{align*}
$$

(3.1e)

$$
\begin{align*}
\dot{x} &= Ax + B_1d + B_2u, \\
y &= C_2x + D_{21}d + D_{22}u, \\
u &= \text{sat}(y_c + v_1),
\end{align*}
$$

(3.1f)

$$
\begin{align*}
\dot{x}_{aw} &= A_{aw}x_{aw} + B_{aw}y_c + E_{aw}r, \\
v_1 &= C_{aw,1}x_{aw} + D_{aw,11}y_c + D_{aw,12}y, \\
v_2 &= C_{aw,2}x_{aw} + D_{aw,21}y_c + D_{aw,22}y,
\end{align*}
$$

(3.1g)

Since $(x, x_c)$ remain the same in the two cls, it is possible to consider only partial coordinate transformations which can be assumed without loss of generality as

$$
\begin{align*}
\xi_M = x_M - T_M x, \\
\xi_F = x_F - T_F x,
\end{align*}
$$

(3.2)

where $x_{aw} = (\xi_M, \xi_F), \xi_M \in \mathbb{R}^n, \xi_F \in \mathbb{R}^{n_F}, T_M \in \mathbb{R}^{n \times n}$ and $T_F \in \mathbb{R}^{n_F \times n}$. By (3.2), $\Sigma_1$ yield

$$
\begin{align*}
\dot{\xi}_M &= A\xi_M + (B_1C_F T_F + AT_M - T_M A)x + B_1C_F \xi_F \\
&+ (B_1D_F - T_M B_1)d + B_2 y_c - T_M B_2 u, \\
\dot{\xi}_F &= A_F \xi_F + (A_F T_F - T_F A)x - T_F B_2 u \\
&+ (B_F - T_F B_1)d, \\
v_1 &= -K\xi_M - K(T_M - I)x, \\
v_2 &= C_2\xi_M + C_2(T_M - I)x,
\end{align*}
$$

and the following answer to questions 1) and 2) above.

**Lemma 3.1.** Systems $\Sigma_1$ and $\Sigma_2$ are equivalent under a partial coordinate transformation (3.2) if and only if there exist real matrices $T_M, T_F, M_1, M_2, M_3, M_4$:

$$
\begin{align*}
M_1C_2 &= B_1C_FT_F + AT_M - T_M A, \quad M_1D_2 = B_1D_F - T_M B_1, \\
M_2C_2 &= A_F T_F - T_F A, \quad M_2D_2 = B_F - T_F B_1, \quad M_3C_2 = K(T_M - I), \quad M_3D_2 = 0, \quad M_4C_2 = C_2(T_M - I), \quad M_4D_2 = 0,
\end{align*}
$$

(3.3)

Given a choice of $F$ and $K$, it is immediate to check if the linear equations (3.3) are satisfied. However, one might wonder how many solutions of (3.3), if any, one can expect to find for different choices of $F$ and $K$. Conditions $M_1D_2 = B_1D_F - T_M B_1$ and $M_2D_2 = B_F - T_F B_1$ alone are very mild (usually, $B_1$ is full column-rank) and easily satisfied (roughly, this guarantees that “no measure of $d$ is really necessary”); however, for generic choices of $F$ and $K$, (3.3) have no solution.

If $n_F = 0$, $D_F = I$ (the $L_2$ aw case), then (3.3b) becomes void, $B_1C_FT_F$ disappears from (3.3a) and the remaining equations are satisfied by $T_M = I$. Clearly, $T_M = I$ satisfies (3.3c)-(3.3d) for any $K$ and $C_2$.

The hard part in (3.3) is given by the Sylvester equation $M_2C_2 = A_F T_F - T_F A$ in (3.3b). If (as usual) $A_F$ and $A$ have no common eigenvalues, there will be a unique solution $T_F$ for any choice of $M_2$, and in general such a solution will not satisfy the condition $M_2D_2 = B_F - T_F B_1$ in (3.3b) for the same value of $M_2$. In fact, (3.3a) and (3.3b) constrain the feasible choices of $F, K$ on a set of measure zero, so that full state measurements are actually needed in order to use generic values of $F, K$. The following corollary of Lemma 3.1 clarifies when the aw compensator in Theorem 2.4 can be implemented by output feedback without introducing additional dynamics (a state estimator). The state
space description of the aw compensator can be easily obtained, so it is omitted for conciseness.

**Theorem 3.2.** Let $F$, $K$ satisfy the conditions in Theorem 2.4. Then, the corresponding output feedback aw compensator with state $x_{aw} \in \mathbb{R}^{n+nf}$ exist iff (3.3) admit a solution $T_M$, $T_F$, $M_1$, $M_2$, $M_3$, $M_4$. □

Coming to question 3), it must be noticed first that since $d$ is an unmeasured input, the asymptotic estimate of $x$ must be obtained by an unknown input observer (UIO). Let $F^-$ denote an arbitrary generalized inverse of $F$ (i.e., a matrix satisfying $FF^-F = F$), and $A := (I - B\hat{C})(A - B_1D_{21}(C_2), B := B_1(I - D_{21}^T D_{21})[C B_1(I - D_{21}^T D_{21})]^T$, $\hat{C} := (I - D_{21}^T D_{21})C_2$. Following [5], the necessary and sufficient conditions for the existence of an UIO for (3.1) are given by the following assumption.

**Assumption 3.3.** $(A, B_1, C_2, D_{21})$ is such that $(A, \hat{C})$ is detectable and $(I - B\hat{C})B_1(I - D_{21}^T D_{21}) = 0$. □

An UIO for (3.1) takes the form:

\[
\dot{\hat{x}} = (A - LC)\theta + B_2u \\
\dot{x} = \theta + BDy
\]  
(3.4a)
(3.4b)

where $D := I - D_{21}^T D_{21}$ and $L$ is chosen such that $(A - LC)$ is Hurwitz. The following theorem extends Theorem 2.4 stating that, provided an UIO for $(A, B_1, C_2, D_{21})$ exists, the same hypotheses of Theorem 2.4 guarantee the existence of an output feedback aw compensator with state $(x_{aw}, \hat{x}) \in \mathbb{R}^{2n+nf}$ solving the problem in Definition 2.3.

**Theorem 3.4.** Let $F$, $K$ satisfy the conditions in Theorem 2.4. Under Assumption 3.3, the aw compensator (2.1) with $x$ replaced by $\hat{x}$ given by (3.4) solves the problem in Definition 2.3. □

The proof of the theorem is similar to the proof of Theorem 2.4. In particular, performing the same partial coordinate transformation used to prove Theorem 2.4, the presence of the observer results in some additional unreachable, exponentially stable dynamics (the error dynamics) generating an $L_2$ disturbance added to $dist_e(\tilde{u}(\cdot))$ at the input of the saturation. The $L_2$ stability in Theorem 2.4 then implies Theorem 3.4.

4. An example

In this section, a simple example is used to illustrate the application of the analysis and the results in this paper. Consider a system described by (3.1), with

\[
A = \begin{bmatrix} -0.1 & 1 \\ -1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} \gamma \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_2 = [1 & 0], \quad D_{21} = D_{22} = 0,
\]

and $D_{21} = D_{22} = 0$, which is supposed to be affected by some unmodelled dynamics of the form

\[
A_F = \begin{bmatrix} -0.6 & 1 \\ -36.09 & 0 \end{bmatrix}, \quad B_F = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}, \quad C_F = [\delta & 0], \quad D_F = 0,
\]

with $\delta \in [-1, 1]$, so that Assumption 2.5 is not satisfied for the a priori given controller

\[
A_c = \begin{bmatrix} -4 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} -40 \\ 80 \end{bmatrix}, \quad C_c = [-1 & 0], \quad D_c = -40,
\]

with $E_c = -B_c$, $F_c = -D_c$. A robustified aw compensator has been designed with filter $(A_F, B_F, C_F, D_F) = (-1, 0.9, 1, 0.1)$ and gain $K = 0$; as the same $F$ can be used also with $K \neq 0$ (possibly leading to better aw performance [7]) compatible with Theorem 2.4, it is now studied when the robustified aw compensator can be implemented by output feedback, possibly for $K \neq 0$.

Both a direct implementation and an UIO based one are considered. Conditions (3.3) for $K = 0$ yield:

\[
T_M = \begin{bmatrix} 0.1 & 0.9 \\ 0.1 & 0.8 \end{bmatrix}, \quad T_F = \begin{bmatrix} 0.9 & 0.9 \\ 1 & 1 \end{bmatrix},
\]

so that the direct implementation is possible for $\gamma \neq -1$. However, if $K \neq 0$ is desired, the additional restriction $K(T_M - I) = M_3C_2$ implies that $K = [K_1 \ K_2]$ must satisfy $K_2 [0.1-0.8s] = 0$, which allows for an arbitrary value of $K_2$ only if $\gamma = 0.125$, whereas $K_2$ is constrained to be zero otherwise. In order to remove the above restriction on the choice of $K$, it is possible to use an UIO under Assumption 3.3; for the considered system, the first condition in the assumption is always true for $\gamma \neq 0$, whilst the second condition requires $s + \gamma^{-1} \neq 0$, $\forall s: \text{Re}[s] \geq 0$, so that the UIO can be adopted iff $\gamma > 0$. For the specific case $\gamma = 0.1 > 0$, both the direct implementation with $K = 0$, and the UIO implementation with $K = [0 \ -100.5]$ (ensuring faster recovery after saturation) are possible. In simulations, in order to show that the aw compensator does not affect the response before saturation occurs, two steps have been applied at $t = 0$s and $t = 25$s, such that saturation occurs only for $t \geq 25$s. The comparison between Fig. 1 and Fig. 2 shows that the robustified aw compensator (equivalent to the $L_2$ aw compensator for $\delta = 0$), has better robustness properties for large parameter deviations ($\delta = 1$). Fig. 3 shows how $K \neq 0$ can improve aw performance (see [7]). Fig. 4 shows (for a much larger reference; notice that, though more slowly, the system output converges to the reference) that the estimation error due to the UIO only causes some exponentially convergent transient.

5. Conclusions

A problem of anti-windup compensation with robustness in the large has been studied in [2], where robusti-
Figure 1: For $\delta = 0$ (nominal parameters), the $L_2$ aw is effective (and the robustified aw coincides with the $L_2$ aw).

Figure 2: For $\delta = 1$, the robustified aw is effective, though the linear cls is unstable (violating Assumption 2.5 of $L_2$ aw).

Figure 3: $K \neq 0$ improves aw performance (faster recovery after saturation).

Figure 4: UIO-based implementation for $K \neq 0$: the estimation error generates small transients, but windup is avoided.

Robustified anti-windup compensators have been proposed; however, such compensators have the drawback of requiring full state measurements. To widen the applicability of the proposed robustified anti-windup compensators, their output feedback implementation has been studied in this paper, giving algebraic conditions (in the form of linear equations) for direct output feedback implementation, and showing that, even if such conditions are not satisfied, the state estimates given by an unknown input observer can be used in place of state measurements. Future work will focus on adaptive tuning of $F$, and the use of multirate unknown input observers (which exist under milder hypotheses then usual unknown input observers).

References