Abstract: The goal of this paper is to propose a family of output feedback compensators (whose state is subject to jumps at the impact times) that stabilize a given $n$ degrees of freedom mechanical system, with linear continuous-time dynamics, subject to non-smooth impacts. An example is used in order to show how the available degrees of freedom can be used to satisfy important requirements, apart from asymptotic stability. Copyright © 2003 IFAC

Keywords: Non-smooth impacts, stabilization, Liapunov method.

1. INTRODUCTION

Mechanical systems subject to impacts occur frequently in every day’s life as well as in the industry. Relevant examples arise in robotics, where the impacts can be either a part of the robot’s task (hopping (Koditschek and Bühler, 1991), walking (Hurmuzlu, 1993; Hurmuzlu et al., n.d.) or juggling (Zunel and Erdmann, 1994; Swanson et al., 1995) robots, tasks like hammering (Izumi and Hitaka, 1997) or inserting a peg into a hole (Prokop and Pfeiffer, 1998)) or, more frequently, undesired collisions, e.g. in space robotics (Nenchev and Yoshida, 1999; Wee and Walker, 1993) or during the so-called “transition phases” between uncostrained motion and a subtask to be executed in permanent contact (Mills and Lokhorst, 1993; Marth et al., 1994).

In all these cases, a suitable model of the impact phenomena is needed; unfortunately the problem of modeling mechanical systems subject to impacts is far from being solved in its generality, especially when multiple impacts or the effects of friction have to be taken into account. In (Tornambè, 1999) two different methods are used to model dynamic systems subject to inequality constraints: the Valentine variables method (which leads to the concept of “non-smooth” impacts) and the penalizing functions method. The efficacy of simple PD control laws for obtaining global asymptotic stability is theoretically proven, for both classes of models, and experimentally tested on a single-link flexible robot arm. Interesting results concerned with the modeling of impacting systems can be found also in (Zheng and Hemami, 1985; Hurmuzlu and Marghitu, 1994; Hurmuzlu and Chang, 1992).

It is important to stress that in most of the existing work on the control of impacting systems...
the impacts are considered “smooth”, i.e., some deformation of the bodies involved is assumed. Here, on the contrary, we consider “non-smooth” impacts, in view of the consideration that, in many cases, the elastic constants which can be used to describe the flexibilities of the impacting bodies are very high, so that it is extremely burdensome to simulate the system’s behavior; for the same reason, the possibility of taking into account the deformations arising during the impacts in the control system design often relies on the availability of very powerful devices for the controller implementation.

In this paper, a family of compensators is proposed for a class of linear mechanical systems subject to a linear inequality constraint, in order to stabilize the origin, which corresponds to a condition of contact, i.e., a configuration in which the constraint is satisfied with the equality sign. Although the mechanical systems considered are linear, when only their unconstrained motion is described, the presence of a linear inequality constraint (and the consequent impulses generated to prevent the possible violation of the constraint at the impact times) renders the system under study nonlinear. The mechanical systems considered have \( n \) degrees of freedom and are fully actuated; only the generalized position coordinates are assumed to be measured, whereas the generalized velocities are not available for measure (this is congruent with the practical impossibility of accurately measuring the velocities in neighborhoods of the impact times). The proposed family of compensators is a modification of the Youla-Kucera parameterization of all stabilizing linear controllers that could be derived for the mechanical system under consideration in absence of the inequality constraint; such a modification (which renders nonlinear also the controllers) is just introduced for taking into account the jumps in the generalized velocities at the impact times. Each one of the compensators belonging to the proposed class guarantees the exponential stability of the origin, and the BIBS (Bounded-Input Bounded-State) stability for a large class of input functions (the latter property is particularly important for impacting mechanical systems).

The work presented here is based on preliminary results reported in (Menini and Tornambè, 2001k; Menini and Tornambè, 2001d; Menini and Tornambè, 2001a; Menini and Tornambè, 2001c).

**Notations.** In the following, \( \mathbb{R} \) will denote the set of real numbers, \( \mathbb{R}^+ \) the set of non-negative real numbers, \( \mathbb{N} \) the set of positive integers, \( \mathbb{R}' \) the set of vectors of dimension \( \nu \) and \( \mathbb{R}'^{\nu \times 
u} \) the set of real matrices of dimensions \( \nu \times \nu \). If \( \mathbf{g} \) is a vector, with \( g_i \) we denote its \( i \)-th entry. Moreover, for a given \( t_0 \in \mathbb{R}, \mathcal{C}^0([t_0, +\infty)) \) will denote the set of all vector functions \( \mathbf{g}(\cdot) \) having entries \( g_{\alpha}(\cdot) : [t_0, +\infty) \rightarrow \mathbb{R} \) that are bounded on every compact subset of \([t_0, +\infty)\). For the sake of brevity, the shorthand notations \( \mathbf{g}(\tau^-) \) and \( \mathbf{g}(\tau^+) \) will be used in place of \( \lim_{t \rightarrow \tau^-} \mathbf{g}(t) \) and \( \lim_{t \rightarrow \tau^+} \mathbf{g}(t) \), respectively, for any vector function \( \mathbf{g}(t) \). At each time \( t = \tau \) at which at least one entry of \( \mathbf{g}(t) \) has a corner point, symbol \( \mathbf{g}(\tau) \) has to be understood as \( \mathbf{g}(\tau^-) \) and \( \mathbf{g}(\tau^+) \). For any matrix \( \mathbf{F} \), \( \mathbf{F}_{\alpha \cdot} \) will denote the \( \alpha \)-th row of \( \mathbf{F} \), \( \mathcal{F}(\mathbf{F}) \) and \( \mathcal{G}(\mathbf{F}) \) will denote the minimum and the maximum singular values of \( \mathbf{F} \), respectively, whereas, if \( \mathbf{F} \) is square, \( \lambda_m(\mathbf{F}) \) and \( \lambda_M(\mathbf{F}) \) will denote the smallest and the largest eigenvalue of \( \mathbf{F} \), respectively. Finally, \( \mathbf{L} \) will denote the \( \nu \times \nu \) identity matrix and the symbol \( \| \cdot \| \) will denote the Euclidean norm of the vector at argument or the corresponding induced norm if the argument is a matrix. In view of this choice, for any matrix \( \mathbf{F} \), we have \( \| \mathbf{F} \| = \mathcal{F}(\mathbf{F}) \).

2. THE CLASS OF CONSIDERED MECHANICAL SYSTEMS

In this paper, we consider \( n \)-degrees of freedom fully actuated linear mechanical systems whose unconstrained motion (i.e., in absence of impacts and contacts) can be described by:

\[
\mathbf{T} \dot{\mathbf{q}}(t) + \mathbf{U}_2 \mathbf{q}(t) + \mathbf{u}_1(t) = \mathbf{\tau}(t). \quad \forall t \geq t_0, (1)
\]

where \( t_0 \in \mathbb{R} \) is the initial time, \( \mathbf{q}(t) \in \mathbb{R}^n \) is the vector of the Lagrangian coordinates, uniquely describing the configuration of the system at time \( t \), \( \mathbf{T} \in \mathbb{R}^{n \times n} \) is the symmetric and positive definite inertia matrix, \( \mathbf{U}_2 \in \mathbb{R}^{n \times n} \) is symmetric, \( \mathbf{u}_1(t) \) is the vector of the (possibly equal to 0, or time-varying) known external forces acting on the system (it can include also potential forces other than \( \mathbf{U}_2 \mathbf{q}(t) \), e.g., the gravity force) and \( \mathbf{\tau}(t) \in \mathbb{R}^n \) is the vector of the control forces. Assume that \( \mathbf{u}_1(\cdot), \mathbf{\tau}(\cdot) \in \mathcal{C}^0([t_0, +\infty)) \). Notice that it is assumed that there is no friction of any kind acting on the system: under standard assumptions, it is easy to extend the theory presented for when the system is subject to dissipative forces proportional to the generalized velocities, whereas nonlinear dependences (as in the case of “stiction”) are to be omitted because they destroy the linearity of (1). The description of the mechanical system under study is completed by considering the following linear inequality constraint:

\[
\mathbf{J} \mathbf{q}(t) \leq 0, \quad \forall t \geq t_0, \quad (\cdot)
\]

where \( \mathbf{J} \in \mathbb{R}^\nu, \mathbf{J} \neq 0 \). A time \( \bar{t} \in \mathbb{R}, \bar{t} > t_0 \), is an impact time if \( \mathbf{J} \mathbf{q}(\bar{t}) = 0 \) and \( \mathbf{J} \mathbf{q}(\bar{t}^-) > 0 \).
However, in the following, especially when only the position \( q(t) \) will be assumed to be measured, it will be difficult to distinguish (in practice) an impact time from a **degenerate impact time**, i.e., a time \( t \in \mathbb{R} \), \( t > t_0 \), such that \( J \mathbf{q}(t) = 0 \), \( J \mathbf{q}(t^-) = 0 \) and there exists \( \varepsilon \in \mathbb{R}, \varepsilon > 0 \), such that \( J \mathbf{q}(t^- - \varepsilon) < 0 \) for all \( \varepsilon \in (0, \varepsilon) \). Hence, for the sake of clarity, from now on we will denote the impact times (degenerate or not) with the notation \( t_i, i \in \mathbb{N} \), ordering them so that \( t_{i+1} > t_i \), for all \( i \in \mathbb{N} \).

We consider the case of a single inequality constraint for simplicity, as it is easy to extend the theory here reported to the case in which more inequality constraints are present, but with more cumbersome notations. In such a case, the only further needed assumption is that no multiple impacts occur, since, to the best of the authors’ knowledge, no satisfactory theory for modeling multiple impacts exists (i.e., by excluding special cases, it is not possible to compute in a unique manner the post-impact velocities after a multiple impact).

**Remark 1.** The previously introduced notation about the impact times actually implies some loss of generality. As a matter of fact, it is not difficult to create examples in which the impact times have one finite accumulation point, followed by other impacts, or even more than one (and, possibly, an infinite number of) finite accumulation points (this occurs especially for inelastic impacts, i.e., with coefficient of restitution \( e < 1 \)). In such a case, assuming that there are no “backward” accumulation points of the impact times, let \( T_j \) be the \( j \)-th accumulation point of the impact times (with \( T_{j+1} > T_j \)); under the assumption that there is no finite accumulation point of the times \( T_j \), let \( T_0 = t_0 \) and let \( t_{j,i} \) be the \( i \)-th impact time after the \((j-1)\)-th accumulation point of the impact times (with \( t_{j,i+1} > t_{j,i} \)). For the sake of simplicity, in this paper it will be always assumed that there is no finite accumulation point of the finite accumulation points of the impact times and also that all the finite accumulation points of the impact times are “forward” accumulation points, i.e., that \( T_j = \lim_{t \to +\infty} t_{j,i} \). Moreover, with a slight abuse of notation, the more complicated notation \( t_{j,i} \) introduced in this remark will be used only in those assumptions and parts of the proofs in which it is actually needed, whereas the proposed results will be stated as if there was no finite accumulation point of the impact times, or at most one finite accumulation point \( T_1 \) but without further impacts in the interval \((T_1, +\infty)\), since these two cases can be dealt with by means of the simpler notation \( t_i \) for the impact times. Nevertheless, the proposed results will be valid in general, with the exception of the cases when there are finite accumulation points of the finite accumulation points of the impact times, or when there are “backwards” accumulation points, which cases are omitted. Finally, the notation \( t_i, i \in \mathbb{N} \), or \( i \in \mathbb{Z}^+ \), will not necessarily imply that the impacts are infinite in number, and the notation \((t_i, t_{i+1})\) has to be understood as \((t_i, +\infty)\) if there are no impact times greater than \( t_i \).

For \( t \in (t_i, t_{i+1}), i \in \mathbb{Z}^+ \), the system is described by the following differential equation:

\[
T \mathbf{q}(t) + U_2 \mathbf{q}(t) + u_1(t) = \mathbf{\tau}(t) + \mathbf{R}(t),
\]

where \( \mathbf{R}(t) \) is the reaction force due to possible contacts, which can be different from zero only in the intervals of time (which necessarily start at degenerate impact times) during which there is a permanent contact. In particular, for \( t \in (t_i, t_{i+1}), i \in \mathbb{Z}^+ \), we have

\[
\mathbf{R}(t) = \begin{cases} 0, & \text{if } (J \mathbf{q}(t) < 0) \lor (J \mathbf{q}(t) = 0 \text{ and } c(t) \leq 0), \\ - \frac{c(t)}{J \mathbf{T}^{-1} J_T} \mathbf{J}_T, & \text{if } J \mathbf{q}(t) = 0 \text{ and } c(t) > 0, \end{cases}
\]

where

\[
c(t) := J_T^{-1} (\mathbf{\tau}(t) - U_2 \mathbf{q}(t) - u_1(t)).
\]

When considering also the impact times \( t = t_i \), \( \mathbf{R}(t) \) includes also the impulsive terms, which guarantee that the inequality constraint \( J \mathbf{q}(t) \leq 0 \) holds for the times \( t \) immediately after \( t_i \); however, it is stressed that, for \( t \in (t_i, t_{i+1}), i \in \mathbb{Z}^+ \), if \( u_1(t) \) and \( \mathbf{\tau}(t) \) are bounded, then \( \mathbf{R}(t) \) is also bounded.

In order to compute the post-impact velocity vector \( \mathbf{q}(t_i) \) as a function of the pre-impact velocity vector \( \mathbf{q}(t_i^-) \), we use the approach described in (Menini and Tornambè, 1999), which is completely equivalent to the **kinetic metric approach** (Brogliato, 1996, Ch. 6). Let \( \mathbf{W} \in \mathbb{R}^{n \times n} \) be a nonsingular matrix that simultaneously diagonalizes \( \mathbf{T} \) and \( \mathbf{J}_T \mathbf{J} \), according to:

\[
\mathbf{W}^T \mathbf{T} \mathbf{W} = \mathbf{I},
\]

\[
\mathbf{W}^T \mathbf{J}_T \mathbf{J} \mathbf{W} = \text{diag}(\mathbf{J} \mathbf{T}^{-1} \mathbf{J}_T^T, 0, \ldots, 0). \tag{6b}
\]

The following lemma, whose proof is rather standard and therefore is omitted here, states the existence of a matrix \( \mathbf{W} \) that satisfies (6) and gives formulae for its computation.

**Lemma 1.** Let \( \mathbf{T} \in \mathbb{R}^{n \times n} \) be symmetric and positive definite, and let \( \mathbf{J}_T \in \mathbb{R}^{n, n} \), \( \mathbf{J} \neq 0 \). Then, there exists a nonsingular matrix \( \mathbf{W} \in \mathbb{R}^{n \times n} \) that satisfies (6), which can be computed as \( \mathbf{W} = \mathbf{H}^{-T} [\mathbf{w}_1 \ldots \mathbf{w}_n] \), where \( \mathbf{H} \in \mathbb{R}^{n \times n} \) is
such that $T = H H^T$, $w_1 := \frac{H^{-1} J^T}{\|H^{-1} J^T\|}$, and \{w_1, w_2, ..., w_n\} is an orthonormal basis for $\mathbb{R}^n$.

Now, let $e \in [0, 1]$ be the coefficient of restitution characterizing the impacts. Let $A(e) \in \mathbb{R}^{n \times n}$ be defined as $A(e) := \text{diag}(-e, 1, ..., 1)$ and $Z(e) := W A(e) W^{-1}$. The rule for computing the post-impact velocity vector is the following:

$$\dot{q}(t_i^+) = Z(e) \dot{q}(t_i^-), \quad i \in \mathbb{N},$$

whereas the position $q(t)$ is a continuous function of time, i.e.: $q(t_i^+) = q(t_i^-)$, $i \in \mathbb{N}$. The mechanical system under consideration is therefore constituted by equation

$$T \dot{q}(t) + U_2 q(t) + u_1(t) = \tau(t) + R(t), \quad t \in (t_i, t_{i+1}), \quad i \in \mathbb{Z}^+,$$

together with (4) and (7), to be solved starting from the initial condition $[q^T(t_0) \: \dot{q}^T(t_0)]^T \in A$. Notice that the presence of the inequality constraint (2) (which is translated into the rule (7)) and the possible presence of a non null $R(t)$ renders the dynamical system under study nonlinear, although (1) is linear.

Rule (7) can be used to justify the choice of the following set of admissible initial conditions:

$$A := \{[q^T \: \dot{q}^T]^T \in \mathbb{R}^{2n}: (J q \leq 0) \text{ and } (J \dot{q} \leq 0 \text{ if } J q = 0)\}.$$

Remark 2. Note that there is no loss of generality in assuming $[q^T(t_0) \: \dot{q}^T(t_0)]^T \in A$, as if $J q(t_0) = 0$ and $J \dot{q}(t_0) > 0$, then $t_0$ is an impact time and it is sufficient to take as initial condition the post-impact state $[q^T(t_0) \: Z(e) \dot{q}^T(t_0)]^T$, which belongs to $A$. \hfill \square

3. THE FAMILY OF PROPOSED COMPENSATORS

In this section, a family of feedback compensators from the measured output $y(t) = q(t)$ is introduced for the mechanical system subject to nonsmooth impacts described by (8) and (7), in order to globally stabilize the origin of the state space. The structure of such a family is closely related to the well-known Youla-Kucera parameterization of all stabilizing controllers for linear time-invariant systems (we followed the approach in (Colaneri et al., 1997), see also (Kucera, 1974; Youla et al., 1976; Francis, 1987; Maciejowski, 1989)). As a matter of fact, the compensators proposed here are described by:

$$\tau(t) = U_2 y(t) + u_1(t) - K_p \dot{q}(t) - K_v \dot{v}(t) + y_Q(t), \quad t \in (t_i, t_{i+1}), \quad i \in \mathbb{Z}^+,$$

$$\dot{q}(t) = \dot{v}(t) + K_1 (y(t) - \dot{q}(t) - r(t)), \quad t \in (t_i, t_{i+1}), \quad i \in \mathbb{Z}^+,$$

$$T \dot{v}(t) = -K_p \dot{q}(t) - K_v \dot{v}(t) + y_Q(t) + R(t) + K_2 (y(t) - \dot{q}(t) - r(t)), \quad t \in (t_i, t_{i+1}), \quad i \in \mathbb{Z}^+,$$

where $y_Q(t) \in \mathbb{R}^m$, $x_Q(t) \in \mathbb{R}^m$, $m \in \mathbb{Z}^+$, $A_Q \in \mathbb{R}^{m \times m}$ has all the eigenvalues with negative real part, $B_Q$, $C_Q$ and $D_Q$ are real matrices of suitable dimensions, such that the pairs $(A_Q, B_Q)$, and $(C_Q, A_Q)$ are reachable and observable, respectively, $R(t)$ is given by (4),

$$K_p := W^{-T} \overline{K}_p W^{-1}, \quad \overline{K}_p := \text{diag}(k_{p,1}, ..., k_{p,n}),$$

$$K_v := \text{diag}(k_{v,1}, ..., k_{v,n}),$$

$$k_{p,h}, k_{v,h} > 0, \text{ for all } h = 1, ..., n, \text{ and } r(t) \in \mathcal{U}^0([t_0, +\infty)) \text{ is the new control signal},$$

$$K_1 := W \text{diag}(k_{1,1}, ..., k_{1,n}) W^{-1}, \quad (9a)$$

$$K_2 := W^{-T} \text{diag}(k_{2,1}, ..., k_{2,n}) W^{-1}, \quad (9b)$$

with $k_{h,j} \in \mathbb{R}$, $k_{h,j} > 0$, $\forall h \in \{1, 2\}$, $\forall j \in \{1, ..., n\}$. It can be seen that if the subsystem having as state vector $x_Q$ is absent, i.e., if $y_Q(t) = 0$, for all $t \geq t_0$, the proposed compensator is a slight modification of the compensator proposed in (Menini and Tornambé, 2001a), whereas, by varying $m$ and the matrices $A_Q$, $B_Q$, $C_Q$ and $D_Q$, apart from the jumps at the impact times, if any, we obtain the family of all stabilizing compensators for the unconstrained system. The compensators described above are nonlinear systems (in view of the jumps imposed to some state variables at the times $t_i$, which are actually functionals of the input signal $y(t)$). However, by limiting our attention to an interval $(t_i, t_{i+1})$, and assuming that $t_i$ is a non-degenerate impact time (so that $R(t) = 0$ for all $t \in (t_i, t_{i+1})$), the control input can be written as $\tau(t) = U_2 y(t) + u_1(t) + \overline{r}(t), \quad \overline{r}(t) = \int_{t_i}^{t} \overline{r}(\tau) \, d\tau$. \hfill \square
where the signal \( \mathbf{y}(t) \) is the output of a linear system having as input \( r(t) - y(t) \) and suitable initial conditions at time \( t_0^+ \). The transfer matrix of such system can be computed as

\[
K(s) = -\left( \mathbf{Y}(s) + \mathbf{M}(s) \mathbf{Q}(s) \right)^{-1} \left( \mathbf{X}(s) + \mathbf{N}(s) \mathbf{Q}(s) \right)^{-1},
\]

where

\[
\begin{align*}
\mathbf{Q}(s) &= \mathbf{D}_Q + \mathbf{C}_Q \left( s \mathbf{I}_n - \mathbf{A}_Q \right) \mathbf{B}_Q, \\
\mathbf{M}(s) &= \left( \mathbf{I}_n + \frac{1}{s} \mathbf{K}_p \mathbf{T}^{-1} + \frac{1}{s^2} \mathbf{K}_v \mathbf{T}^{-1} \right)^{-1}, \\
\mathbf{N}(s) &= \Delta^{-1}(s), \\
\mathbf{X}(s) &= \mathbf{I}_n + \frac{1}{s} \Delta^{-1}(s) \cdot \\
& \quad \left( s \mathbf{I}_n + \mathbf{T}^{-1} \mathbf{K}_v \right) \mathbf{K}_1 + \mathbf{T}^{-1} \mathbf{K}_2, \\
\mathbf{Y}(s) &= -\frac{1}{s} \mathbf{K}_v \Delta^{-1}(s) \cdot \\
& \quad \left( s \mathbf{I}_n + \mathbf{T}^{-1} \mathbf{K}_v \right) \mathbf{K}_1 + \frac{1}{s} \mathbf{T}^{-1} \mathbf{K}_2 + \mathbf{K}_p \Delta^{-1}(s) \left( \frac{1}{s} \mathbf{T}^{-1} \mathbf{K}_p \mathbf{K}_1 - \mathbf{T}^{-1} \mathbf{K}_2 \right),
\end{align*}
\]

with \( \Delta(s) = s \mathbf{I}_n + \mathbf{T}^{-1} \mathbf{K}_v + \frac{1}{s} \mathbf{T}^{-1} \mathbf{K}_p \).

The closed-loop system can be rewritten as follows, using the variables \( \mathbf{v} = \mathbf{q}, \mathbf{q} = \mathbf{q} - \mathbf{q} \) and \( \mathbf{v} = \mathbf{v} - \mathbf{v} \):

\[
\begin{align*}
\quad \dot{\mathbf{q}}(t) &= \mathbf{v}(t), & t \in (t_i, t_{i+1}), & i \in \mathbb{Z}^+, & (15a) \\
\quad \mathbf{T} \dot{\mathbf{v}}(t) &= -\mathbf{K}_p \left( \mathbf{q}(t) - \mathbf{q}(t) \right) + \mathbf{D}_Q \dot{\mathbf{q}}(t) \\
& \quad - \mathbf{K}_v \left( \mathbf{v}(t) - \mathbf{v}(t) \right) + \mathbf{C}_Q \mathbf{x}_Q(t) - \\
& \quad \mathbf{D}_Q \mathbf{r}(t) + \mathbf{R}(t), & t \in (t_i, t_{i+1}), & i \in \mathbb{Z}^+, & (15b) \\
\quad \dot{\mathbf{q}}(t) &= \mathbf{v}(t) - \mathbf{K}_1 \mathbf{q}(t) + \mathbf{K}_1 \mathbf{r}(t), & t \in (t_i, t_{i+1}), & i \in \mathbb{Z}^+, & (15c) \\
\quad \mathbf{T} \dot{\mathbf{v}}(t) &= -\mathbf{K}_2 \dot{\mathbf{q}}(t) + \mathbf{K}_2 \mathbf{r}(t), & t \in (t_i, t_{i+1}), & i \in \mathbb{Z}^+, & (15d) \\
\quad \mathbf{x}_Q(t) &= \mathbf{A}_Q \mathbf{x}_Q(t) + \mathbf{B}_Q \mathbf{q}(t) - \mathbf{B}_Q \mathbf{r}(t), & t \in (t_i, t_{i+1}), & i \in \mathbb{Z}^+, & (15e) \\
\quad \mathbf{v}(t_i^+) &= \mathbf{Z}(\mathbf{e}) \mathbf{v}(t_i^-), & i \in \mathbb{N}, & (15f) \\
\quad \dot{\mathbf{q}}(t_i^+) &= \mathbf{Z}(\mathbf{e}) \dot{\mathbf{q}}(t_i^-), & i \in \mathbb{N}, & (15g) \\
\quad \mathbf{v}(t_i^+) &= \mathbf{Z}(\mathbf{e}) \mathbf{v}(t_i^-), & i \in \mathbb{N}. & (15h)
\end{align*}
\]

Denote by

\[
\mathbf{x}_{cc}(t) = \begin{bmatrix} \mathbf{q}^T(t) & \mathbf{v}^T(t) & \mathbf{q}^T(t) & \mathbf{v}^T(t) & \mathbf{x}_Q^T \end{bmatrix}^T
\]

the state of the system (15).

Assumption 1. Let \( \mathbf{x}_{cc}(t_0) \in \mathcal{A} \times \mathbb{R}^{2n+m} \) and \( \mathbf{r}(\cdot) \in \mathcal{C}^0([t_0, +\infty)) \) be such that the solution of the closed-loop system (15) has no finite accumulation points of the impact times.

Theorem 1. Let \( e \in [0, 1] \). If \( k_{p,j} > 0, k_{v,j} > 0, k_{h,j} > 0, \) \( \forall j \in \{1, \ldots, n\} \), and all the eigenvalues of \( \mathbf{A}_Q \) have negative real part, then

(i) the equilibrium \( \mathbf{x}_{cc} = 0 \) of system (15) is a globally exponentially stable equilibrium with respect to all the initial states satisfying Assumption 1 together with \( \mathbf{r}(\cdot) = 0 \), i.e., there exist \( \alpha > 0 \) and \( \beta > 0 \) such that, if \( \mathbf{x}_{cc}(t_0) \) and \( \mathbf{r}(\cdot) = 0 \) satisfy Assumption 1, then the following relation holds for \( \mathbf{r}(\cdot) = 0 \):

\[
\|\mathbf{x}_{cc}(t)\| \leq \alpha \exp(-\beta(t-t_0)) \|\mathbf{x}_{cc}(t_0)\|, \\
\forall t \geq t_0, \forall t \geq t_0; \tag{16}
\]

(ii) the closed-loop system (15) is globally bounded-input bounded-state (shortly, BIBS) stable for all the initial states and inputs satisfying Assumption 1, i.e., there exist two functions \( \gamma_x(\cdot, \cdot) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \) and \( \gamma_r(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+ \), both strictly increasing with respect to the first argument, \( \gamma_x(s, z) = 0 \) for every \( s > 0 \), and such that \( \gamma_x(0, z) = 0 \), for all \( z \in \mathbb{R}^+ \) and \( \gamma_x(0, 0) = 0 \), such that, if \( \mathbf{x}_{cc}(t_0) \) and \( \mathbf{r}(\cdot) \) satisfy Assumption 1 and \( \|\mathbf{r}(t)\| \leq r_M \), \( r_M \in \mathbb{R}^+ \), for all \( t \geq t_0 \), then the solution \( \mathbf{x}_{cc}(\cdot) \) of system (15) satisfies:

\[
\|\mathbf{x}_{cc}(t)\| \leq \gamma_x(\|\mathbf{x}_{cc}(t_0)\|, t-t_0) + \gamma_r(r_M), \\
\forall t \geq t_0. \tag{17}
\]

Remark 3. One of the nicest properties of the Youla-Kucera parameterization for linear time-invariant systems is that it allows to represent by means of the parameter \( \mathbf{Q}(s) \) the class of all linear stabilizing compensators for the given system. Theorem 1 states a weaker property of the family of compensators proposed here for linear mechanical systems subject to impacts, since it only states that each compensator in the family is a stabilizing one, but it does not exclude the existence of other stabilizing compensators. This is actually due to the presence of the inequality constraint, which renders the system nonlinear, and (in some situations) can exert an additional “stabilizing” action. As an example, in the case of \( e = 1 \) (no dissipation occurs at the impact times), consider the constrained one-degree of freedom system described by \( \mathbf{q} = \tau, \mathbf{q} < 0 \), and the control law \( \tau = 9.81 - k_v \mathbf{q} \), with \( k_v > 0 \). It can be easily seen that the origin \( \mathbf{q} = 0 \), \( \dot{\mathbf{q}} = 0 \) is an asymptotically stable equilibrium point for the closed-loop system in presence of the inequality constraint (it is the model of a mass subject to viscous friction, bouncing on the ground under the action of the gravity force), whereas it is not (it is not even an equilibrium) in absence of the constraint. \( \square \)
Proof of Theorem 1. By rewriting system (15) in the variables \( z = W^{-1} q, \ \dot{v} = W^{-1} \dot{q}, \ \tilde{z} = W^{-1} \tilde{q} \) and \( v = W^{-1} \tilde{v} \), where \( \tilde{q}(t) := q(t) - q(t) \) and \( \tilde{v}(t) := v(t) - v(t) \) we obtain:

\[
\begin{align*}
\dot{z}(t) &= v_z(t), \quad t \in (t_i, t_{i+1}), \ i \in \mathbb{Z}^+, \\
\dot{v}_z(t) &= -K_P (z(t) - \tilde{z}(t)) + \bar{D}_Q \dot{z}(t) - \bar{K}_r (v_z(t) - \tilde{v}_z(t)) + C_Q x_Q(t) - \bar{D}_Q r(t) + R_z(t), \\
&\quad t \in (t_i, t_{i+1}), \ i \in \mathbb{Z}^+, \\
\tilde{z}(t) &= v_z(t) - \text{diag}(k_{1,1}, ..., k_{1,n}) \tilde{z}(t) + W^T K_Z r(t), \\
&\quad t \in (t_i, t_{i+1}), \ i \in \mathbb{Z}^+, \\
x_Q(t) &= A_Q x_Q(t) + B_Q \tilde{z}(t) - B_Q r(t), \\
&\quad t \in (t_i, t_{i+1}), \ i \in \mathbb{Z}^+, \\
\end{align*}
\]

Due to the fact that, if \( R_z(t) = 0 \) then \( Z(t) = 0 \) and \( V_z(t) = 0 \), it is easy to see that equations (18a)-(18e) can be rewritten in compact form in terms of the state vector \( x_{zc} \), as follows:

\[
\begin{align*}
x_{zc}(t) &= \begin{cases}
A_{zc} x_{zc}(t) + B_{zc} r(t), & \text{if } R_z(t) = 0, \\
A_{zc} x_{zc}(t) + B_{zc} r(t), & \text{if } R_z(t) \neq 0,
\end{cases}
\end{align*}
\]

where the only difference between the matrices \( A_{zc} \) and \( A_{zc} \) (between the matrices \( B_{zc} \) and \( B_{zc} \)) is that the components in positions \( (2, 2n + 1), ..., (2, 4n + m) \) of matrix \( A_{zc} \) are zero (the second row of matrix \( B_{zc} \) is zero), whereas the corresponding components of \( A_{zc} \) (the second row of \( B_{zc} \)) can be different from zero, in general. Let

\[
\begin{align*}
A_z := & \text{blockdiag} \left( \begin{bmatrix} 0 & 1 \\ -k_{p,1} & -k_{v,1} \end{bmatrix}, ..., \begin{bmatrix} 0 & 1 \\ -k_{p,n} & -k_{v,n} \end{bmatrix} \right), \\
B_z := & \text{blockdiag} \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix}, ..., \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right), \\
Q_z := & \text{blockdiag} \left( \begin{bmatrix} -k_{1,1} \\ 0 \end{bmatrix}, ..., \begin{bmatrix} -k_{1,n} \\ 0 \end{bmatrix} \right).
\end{align*}
\]

Consider the quadratic function \( V_{zc} = x_{zc}^T P_{zc} x_{zc} \), where \( P_{zc} = \text{blockdiag} (P_z, P_o, \eta P_Q) \), with \( P_z \), \( P_o \) and \( P_Q \) are, respectively, the solutions of the following Lyapunov equations

\[
\begin{align*}
P_z A_z + A_z^T P_z &= -I_{2n}, \\
P_o A_o + A_o^T P_o &= -I_{2n}, \\
P_Q A_Q + A_Q^T P_Q &= -I_{m},
\end{align*}
\]

and \( \varepsilon \) and \( \eta \) are positive parameters, whose values will be fixed later. By computing the matrices \( Q_{zc} = -(P_{zc} A_{zc} + A_{zc}^T P_{zc}) \),

\[
\begin{align*}
Q_{zc} &= -(P_{zc} A_{zc} + A_{zc}^T P_{zc}), \\
\bar{Q}_{zc} &= -(\tilde{P}_{zc} \tilde{A}_{zc} + \tilde{A}_{zc}^T \tilde{P}_{zc}),
\end{align*}
\]

in view of the block diagonal form of \( P_{zc} \), it can be seen that

\[
\begin{align*}
Q_{zc} &= \begin{bmatrix}
I_{2n} & Q_1 & Q_2 \\
Q_1^T & \varepsilon I_{2n} & \eta Q_3 \\
Q_2^T & \eta Q_3^T & \eta I_m
\end{bmatrix}, \\
\bar{Q}_{zc} &= \begin{bmatrix}
I_{2n} & \tilde{Q}_1 & \tilde{Q}_2 \\
\tilde{Q}_1^T & \varepsilon I_{2n} & \eta \tilde{Q}_3 \\
\tilde{Q}_2^T & \eta \tilde{Q}_3^T & \eta I_m
\end{bmatrix},
\end{align*}
\]

where \( \tilde{Q}_1 \) and \( \tilde{Q}_2 \) are equal to \( Q_1 \) and \( Q_2 \), respectively, apart from the first two rows of \( \tilde{Q}_1 \) and \( \tilde{Q}_2 \), which are always null, whereas the first two rows of \( Q_1 \) and \( Q_2 \) need not be null. Then,
letting \( \chi_h = \|Q_h\|, h = 1, 2, 3, \) if \( r(\cdot) = 0, \) we have
\[
\dot{V}_{zc} \leq - \left[ \|x_c\| \|\dot{x}_c\| \|x_Q\| \right] \hat{Q}_{zc} \left[ \|\dot{x}_c\| \|\dot{x}_c\| \|x_Q\| \right], \quad \forall t \geq t_0,
\]
where
\[
\hat{Q}_{zc} = \begin{bmatrix}
1 & -\chi_1 & -\chi_2 \\
-\chi_1 & \varepsilon & -\eta \chi_3 \\
-\chi_2 & -\eta \chi_3 & \eta
\end{bmatrix}.
\]
By computing the three leading principal minors of \( \hat{Q}_{zc}: \)
\[
\{1, \varepsilon - \chi_1^2, \\
\varepsilon \eta - \eta^2 \chi_3^2 - \eta \chi_1^2 - 2 \eta \chi_1 \chi_2 \chi_3 - \chi_3 \varepsilon^2\},
\]
it can be seen that the choices \( \eta = 2 \chi_3^2 \) (the fact that the pair \((C_Q, A_Q)\) is observable implies that \(C_Q\) cannot be zero, and this implies that \( \chi_2 > 0 \)) and \( \varepsilon > \varepsilon^*, \) with \( \varepsilon^* := 4 \chi_3^2 \chi_2^2 + 2 \chi_1^2 + 4 \chi_1 \chi_2 \chi_3, \) render matrix \( \hat{Q}_{zc} \) (and therefore both matrices \( Q_{zc} \) and \( Q_{ze} \)) positive definite. Whence, with such choices, \( V_{zc} \) is a Liapunov function for system (18). Let \( \xi_e := \min \{\lambda_{m}(Q_{zc}), \lambda_{m}(Q_{ze})\}. \)

By computations wholly similar to the ones used in (Menini and Tornambe, 2001b), we can write
\[
\|x_{zc}(t)\| \leq \frac{S_M}{S_m} \sqrt{\frac{\lambda_{M}(P_{zc})}{\lambda_{m}(P_{zc})}} \|x_{zc}(t_0)\| \cdot \exp \left( -\frac{\xi_e}{2 \lambda_{M}(P_{zc})} (t - t_0) \right),
\]

which is relation (16) with \( \alpha := \frac{S_M}{S_m} \sqrt{\frac{\lambda_{M}(P_{zc})}{\lambda_{m}(P_{zc})}} \) and \( \beta := \frac{\xi_e}{2 \lambda_{M}(P_{zc})}. \) As for assertion (ii), again using computations wholly similar to the ones in (Menini and Tornambe, 2001b), if \( \|r(t)\| < \tau_M, \) \( \forall t \geq t_0, \) we can write
\[
\|x_{zc}(t)\| \leq \sqrt{\frac{\lambda_{M}(P_{zc})}{\lambda_{m}(P_{zc})}} \|x_{zc}(t_0)\| \cdot \exp \left( -\frac{\xi_e}{2 \lambda_{M}(P_{zc})} (t - t_0) \right) + \frac{2 \chi_3^2 \lambda_{M}(P_{zc})}{\lambda_{m}(P_{zc})} \xi_e \|B_{zc}\| \|W\| \tau_M,
\]
from which, by (19), we derive that relation (17) holds with
\[
\gamma_x(s, z) := \frac{S_M}{S_m} \sqrt{\frac{\lambda_{M}(P_{zc})}{\lambda_{m}(P_{zc})}} \exp \left( -\frac{\xi_e}{2 \lambda_{M}(P_{zc})} \right) s, \\
\gamma_r(s) := \frac{2 \sigma(W) \lambda_{M}(P_{zc})}{S_m \lambda_{m}(P_{zc})} \xi_e \|B_{zc}\| s.
\]

The example described in next section shows how the parameterization of a family of stabilizing compensators can be used to deal with control problems involving different requirements in addition to asymptotic and BIBS stability.

4. AN EXAMPLE OF APPLICATION

Consider a dimensionless body having mass \( M \) that moves (without friction) on a vertical plane, subject to the gravity acceleration, of magnitude \( g. \) Letting \( q_1(t) \) and \( q_2(t) \) denote the horizontal and vertical position coordinates of the body at time \( t, \) respectively, and assuming that it is possible to exert a horizontal and a vertical force on the body, which are denoted by \( r_1(t) \) and \( r_2(t), \) the equations of motion for the unconstrained body are:
\[
\dot{q}_1(t) = r_1(t)/M, \\
\dot{q}_2(t) = r_2(t)/M - g.
\]

Let the body be constrained to move on the upper half plane delimited by \( q_2(t) \geq q_1(t), \) so that, in our setting, \( J = \begin{bmatrix} 1 & 0 \end{bmatrix}, \) and let \( e = 1; \) it can be computed that, at the impact times, we have \( \dot{q}_1(t^+_n) = \dot{q}_2(t^+_n) \) and \( \dot{q}_2(t^-_n) = \dot{q}_1(t^-_n). \) Let only the position coordinates \( q(t) := [q_1(t) q_2(t)]^T \) be available for measurement. We assume, further, that the mass of the body is not exactly known, and only its nominal value \( M_0 \) is available for control design. It is clear that the goal of regulating the position of the body to the origin \( q = 0 \) cannot be obtained robustly with respect to the mass variation by means of the PD compensator from the observed state proposed in (Menini and Tornambe, 2001a) (which is practically the one proposed here with the “Q” subcompensator set equal to zero), since the compensation of the gravity acceleration is not exact if \( M \neq M_0. \) A first set of simulations has been carried out to confirm this fact, by using the output feedback compensator proposed in (Menini and Tornambe, 2001a), with gains \( k_{p,1} = k_{p,2} = 3, k_{v,1} = k_{v,2} = 4, k_{1,1} = k_{1,2} = 15 \) and \( k_{2,1} = k_{2,2} = 50. \) In Figure 1, the results of three of such simulations, corresponding to \( M_0 = 1 \) kg (the nominal case, represented by the bold lines), \( M = M_1 = 0.95 \) kg (represented by the dashed lines) and to \( M = M_2 = 1.1 \) kg (represented by the continuous lines), respectively, are reported. In the plots (b) and (c), horizontal lines represent the equilibrium positions of the constrained closed-loop system. In both the perturbed cases, since the gravity force is not compensated exactly, such equilibrium positions are different from the origin. Notice that, for \( M = M_1 \) (dashed lines), such an equilibrium coincides with the one of the unconstrained system (it is not a contact configuration), whereas, for \( M = M_2, \) the equilibrium of the unconstrained system does not belong to the admissible region.
thus implying that the system experiences an infinite number of impacts approaching the only equilibrium of the constrained system.

In order to solve the mentioned robust regulation problem, the degrees of freedom for control design provided by the parameterization proposed in this section can be used. Observe, first, that the system to be controlled is constituted, if the constraint is neglected, by two SISO decoupled subsystems, having transfer function $q_h(s)/r_h(s) = 1/(M s^2)$, $h = 1, 2$, which interact only at the impact times. Therefore, it is very easy to design, for each of such subsystems, a linear time invariant compensator which solves the robust regulation problem, just by modelling the wrong compensation of the gravity acceleration as an additional constant input disturbance, to be asymptotically rejected. To this end, two identical linear time invariant compensators having transfer functions $r_h(s)/(q_h(s) - r_h(s)) = C_h(s) = (-b_2 s^2 + b_1 s + b_0)/(s^2 + a_1 s)$. $h = 1, 2$, have been designed. by placing the closed-loop poles of the nominal system at the same locations obtained by the PD compensator mentioned above. Then, by using the formulae given in (Colaneri et al., 1997, Sec. 3.4), such compensators have been transformed into the connection of the PD compensator used above with the “Q” subsystem, which turns out to be a suitable static gain. As a final step, the jumps (15g) and (15h) have been incorporated in the compensator. The results of the three simulations of the closed-loop systems corresponding to $M = M_0$, $M = M_1$ and $M = M_2$ (with the same initial conditions of the simulations in Figure 1) are reported in Figure 2. It can be appreciated that regulation is obtained in all three cases. In addition, since two identical compensators have been used for the “horizontal” and “vertical” subsystems, asymptotic rejection of constant input disturbances is obtained with such a compensator.

5. CONCLUDING REMARKS

The strongest assumption made in this paper is the linearity of the mechanical system when undisturbed by the impacts. This assumption allows one to write in closed form a family of compensators guaranteeing for the closed-loop system both the Liapunov stability and the BIBS stability; if the unconstrained system is nonlinear, then the compensators proposed can be applied only locally. It seems difficult to overcome the mentioned assumption, since, also when the mechanical system is unconstrained, it is hard (if not impossible) to design the family of all globally stabilizing controllers.
Future work will be devoted to study choices of the free parameter $Q(s)$ so that we can guarantee additional properties for the closed-loop system, such as parametric or non-parametric robustness, asymptotic or practical trajectory tracking (possibly, involving a large number of or an infinite number of impacts), asymptotic or practical disturbance rejection (possibly guaranteeing a condition of permanent contact also in the presence of disturbances).

REFERENCES


