Semi-Active Control of a Thin Piezoactuated Structure

Paolo Bisegna\textsuperscript{a}, Giovanni Caruso\textsuperscript{a}, Dionisio Del Vescovo\textsuperscript{b}, Sergio Galeani\textsuperscript{c}, and Laura Menini\textsuperscript{c}

\textsuperscript{a} Dept. Civil Engineering, Univ. Rome "Tor Vergata", 00133 Rome, Italy
\textsuperscript{b} Dept. Mechanics and Aeronautics, Univ. Rome "La Sapienza", 00186 Rome, Italy
\textsuperscript{c} Dept. Informatics, Systems and Production, Univ. Rome "Tor Vergata", 00133 Rome, Italy

ABSTRACT

Vibration damping of a cantilever plate is achieved by using a piezoelectric element simultaneously as passive single-mode device and active broad-band actuator. Control strategies are designed on the basis of a modal model of the coupled electro-mechanical structure. This model is obtained by using a suitable finite-element formulation together with a modal analysis. A purely passive single-mode control composed of an optimally tuned external RL shunt circuit and a purely active control based on classical LQG techniques are compared to a semi-active control obtained by tuning the external shunt circuit on the second vibration mode of the structure and using a LQG controller designed on the only first-mode model.

Keywords: Vibration suppression, LQG control, damped piezoelectric actuation, piezoelectric finite-elements

1. INTRODUCTION

Intelligent technologies are currently employed in civil, aeronautic and space structures. Among their numerous applications, vibration suppression of structures is analyzed in this work. Vibration damping can be achieved by using piezoelectric actuators, due to their capability to drain mechanical energy from the structure and to convert it into electric energy, which can be, in turn, easily dissipated through a passive or active electric network.

Of course, bonding piezoelectric actuators on the vibrating structure makes a coupled electro-mechanical behaviour arise. Needless to say that satisfactory vibration control strategies must be based on a reliable modelization of the coupled vibrating structure.

In this paper a vibrating plate structure having thin piezoelectric actuators bonded on its upper and/or lower surfaces is analyzed. Its geometrical details and material properties are reported in Section 2.4. The structure is modeled according to a finite-element formulation recently proposed by Bisegna and Caruso\textsuperscript{1}: the underlying variational formulation is briefly recalled in Section 2.1. The adopted finite element is two-dimensional, quadrangular, four-node, Mindlin-type, locking-free, and has four degrees of freedom per node (i.e., the deflection of the middle plane of the plate, the rotations of the fibers normal to the middle plane and the actuation electric potential of the piezoelectric actuator). It is used to build a discrete model for the bending vibration of the structure, presented in Section 2.2. A modal analysis is then performed in Section 2.3. A modal reduction is then applied, in order to obtain a small-size discretized model useful in vibration control design. The capability of the proposed approach of generating accurate low- and high-order models is very valuable, since low-order models can be used to design a controller, whereas high-order models allow to evaluate the actual performance of the controller when applied to the real system.

It is well known that piezoelectric elements can act both as passive single-mode devices and as active broadband actuators. A passive single-mode device is obtained shunting the piezoelectric actuator to an external electric RL circuit, tuned to the structural natural frequency to be suppressed.\textsuperscript{2} The attainable modal damping ratio strongly depends on the values of the electric components $R$ and $L$. The problem of finding optimal electric parameters was addressed and solved by Hagood and von Flotow,\textsuperscript{2} under the hypothesis of negligible structural damping of the vibrating structure. Hagood and von Flotow's formulas are reported in Section 3, and generalized to the more usual situation of a nonvanishing structural damping.

Further author information: (Send correspondence to P.B.)

P.B.: E-mail: bisegna@ing.uniroma2.it
L.M.: E-mail: menini@disp.uniroma2.it
A multimode piezoelectric passive damper can be obtained by adding parallel additional resistor-inductor-capacitor branches for each additional mode to be damped. However, passive piezoelectric devices cannot provide broadband response control required by many applications. On the other hand, a broadband response control can be obtained by imposing either a feedback control voltage or current on the piezoelectric electrodes. In this paper it is assumed that the sensing signal is provided by an accelerometer close to the free edge of the cantilever plate. In Section 4 a purely active control based on classical LQG techniques is presented. Special care is used in order to reduce spillover effects, which of course constitute the main concern with active control. In particular, once a satisfactory controller is found for a reduced-order model of the system, its performance is evaluated by simulation, using a higher-order model. The quadratic indexes used in the design process are chosen here according to the algorithm by Kraus and Kučera, in order to properly change the eigenvalues of the reduced-order closed-loop system, thus obtaining a better performance on the higher-order model.

Most works have avoided spillover problems using collocated self-sensing actuators: that approach guarantees stability for a purely proportional output rate feedback, at least as far as ideal actuators are concerned. As a matter of fact, even small deviations from the ideal situation may lead to a loss of observability, controllability or stability of the system. In this paper a completely different approach to the spillover problem is adopted: an additional passive damping is added to the system in order to achieve higher robustness and stability margins. This reduces the chance of control spillover that can destabilize the system. As suggested by Agnes, the same piezoelectric actuator is used here as both passive resonance suppression device and broadband actuator. This kind of control is referred here as semi-active control.

A robust and performant compensator is presented in Section 5. It is assumed that the external RL circuit is tuned in order to damp the second natural vibration mode of the structure. The dynamic compensator is designed on a reduced-order two-degrees-of-freedom model of the electro-mechanical system, involving the first natural mechanical vibration mode of the plate and the RL electric circuit.

The performances of both the purely active control law and the semi-active one are evaluated in simulation by comparing the results they supply with the ones obtained by means of purely passive control.

2. COUPLED ELECTRO-MECHANICAL STRUCTURE

This Section is devoted to the presentation of a suitable modelization of the the coupled electro-mechanical system shown in Figure 1.

2.1. Variational formulation

A linearly-elastic, isotropic, homogeneous plate is considered, with Young modulus $E$ and Poisson ratio $\nu$. Let $\Omega$ be the middle cross-section of the plate. A Cartesian frame is chosen as shown in Figure 1. A transversely-isotropic, linearly-piezoelectric, homogeneous actuator is bonded on the upper surface of the plate, as shown in Figure 1. The piezoelectric constitutive behavior is completely described by ten closed-circuit/clamped material constants, denoted by $c_{11}^p$, $c_{33}^p$, $c_{44}^p$, $c_{12}^p$, $c_{13}^p$, $c_{11}^p$, $c_{33}^p$, $c_{31}^p$, $c_{33}^p$ and $e_{16}^p$, according to the classical Voigt notation. Here the superscript $p$ is used to distinguish any quantity relevant to the piezoelectric actuator.
In addition, the following material constants, relevant to the situation of negligible transversal stress inside the piezoelectric actuator, are introduced:

\[
\begin{align*}
\varepsilon_{33}^p &= \varepsilon_{33} + (\varepsilon_{33}^p)^2 / \varepsilon_{33}^p \\
\varepsilon_{11}^p &= \varepsilon_{11} + (\varepsilon_{11}^p)^2 / \varepsilon_{11}^p \\
\varepsilon_{12}^p &= \varepsilon_{12} - (\varepsilon_{12}^p)^2 / \varepsilon_{12}^p \\
\varepsilon_{31}^p &= \varepsilon_{31} - \varepsilon_{31}^p / \varepsilon_{33}^p \\
\nu^p &= \varepsilon_{12}^p / \varepsilon_{11}^p
\end{align*}
\]

A variational formulation for the coupled electro-mechanical system is developed under the following simplifying assumptions:

i) the thickness \( t^p \) of the piezoelectric actuator is negligible with respect to the thickness \( t \) of the plate. Thus, only the in-plane (or membranal) behavior of the piezoelectric actuator is taken into account;

ii) the in-plane displacement of the piezoelectric actuator is constant inside the thickness of the actuator, and is denoted by \( u^p = (u_1^p, u_2^p) \);

iii) the in-plane stiffness of the piezoelectric actuator is negligible with respect to the in-plane stiffness of the plate;

iv) the displacement field \((s_1, s_2, s_3)\) of the plate is represented according to Mindlin's hypotheses:

\[
\begin{align*}
S_1(x_1, x_2, x_3) &= \varphi_1(x_1, x_2) x_3 \\
S_2(x_1, x_2, x_3) &= \varphi_2(x_1, x_2) x_3 \\
S_3(x_1, x_2, x_3) &= w(x_1, x_2)
\end{align*}
\]

where \( w \) is the deflection of the middle plane of the plate and \( \varphi = (\varphi_1, \varphi_2) \) is the rotation of the fibers parallel to \( x_3 \);

v) the electric potential \( v^p \) is linear across the thickness of the piezoelectric actuator and hence the electric field along the \( x_3 \) axis can be expressed as \(-v^p / t^p\);

vi) the transversal stress \( \sigma_{33} \) in the plate and in the piezoelectric actuator is negligible.

Under the previous assumptions, the electro-mechanical potential energy \( E \) of the piezoelectric laminate may be considerably simplified and turns out to be:

\[
E = E_m + E_{m}^p + E_{e}^p + E_{em} + U_m + U_{e}^p
\]

where different terms may be recognized:

- \( E_m \) is the elastic potential energy of the plate:

\[
E_m = \frac{Et^3}{24(1 - \nu^2)} \int_{\Omega} [(1 - \nu)(|\nabla \varphi|^2 + \nu(\text{div} \varphi)^2) \, da + \frac{tk\mu}{2} \int_{\Omega} ||\varphi + \nabla w||^2 \, da
\]

where the first integral is the bending energy and the second integral is the shear energy; moreover, \( \mu = E/[2(1 + \nu)] \) is the shear modulus, \( k = 5/6 \) is the shear factor, \( || \cdot || \) denotes the norm, \( \nabla \) and \( \text{div} \) are, respectively, the gradient and the divergence operators with respect to the \( x_1, x_2 \) variables and a hat denotes the symmetric part of a tensor.

- \( E_{m}^p \) is the elastic potential energy in the piezoelectric layer:

\[
E_{m}^p = \frac{t^p \varepsilon_{11}^p}{2} \int_{\Omega} [(1 - \nu^p)(|\nabla u^p|^2 + \nu^p(\text{div} u^p)^2) \, da
\]

- \( E_{e}^p \) is the electrostatic potential energy in the piezoelectric layer:

\[
E_{e}^p = \frac{\varepsilon_{33}^p}{2\mu} \int_{\Omega} (v^p)^2 \, da - \frac{\varepsilon_{11}^p}{24} \int_{\Omega} ||\nabla v^p||^2 \, da
\]

where the first integral takes into account the energy associated with electric field along \( x_3 \) and the second integral is due to the in-plane electric field.
• $E_{em}$ is the electro-mechanical coupling potential energy:

$$E_{em} = \varepsilon_{11} \int_{\Omega} v^p \text{div} u^p \, da \quad (8)$$

• $U_m$ is the potential energy of the external load $q$, normal to the plate:

$$U_m = - \int_{\Omega} q w \, da \quad (9)$$

• $U^p_e$ is the potential energy of the free electric charge $\omega$ on the piezoelectric surfaces:

$$U^p_e = \int_{\Omega} \omega v^p \, da \quad (10)$$

The in-plane displacement $u^p$ of the piezoelectric layer is given by $u^p = \frac{t}{2} \varphi$, due to the requirement of continuous displacements through the thickness of the structure.

In order to perform a dynamical analysis of the coupled system, it is necessary to evaluate also its kinetic energy $T = T_m + T^p_m$, where the following terms can be distinguished:

• $T_m$ is the kinetic energy of the plate:

$$T_m = \frac{\rho}{2} \int_{\Omega} [t \dot{w}^2 + \frac{t^3}{12}||\varphi||^2] \, da \quad (11)$$

• $T^p_m$ is the kinetic energy of the piezoelectric actuator:

$$T^p_m = \frac{\rho^p}{2} \int_{\Omega} [t^p \dot{\varphi}^2 + \frac{t^2 t^p}{4}||\varphi||^2] \, da \quad (12)$$

where $\rho$ and $\rho^p$ are, respectively, the densities of the material comprising the plate and the actuator, and a dot denotes differentiation with respect to the time.

The equations governing the dynamic behaviour of the actuated vibrating plate can be easily obtained by using the Hamilton principle. They are not reported here for the sake of brevity.

2.2. Finite-element model

From the foregoing variational formulation a finite-element formulation can be obtained. It is based on a two-dimensional, quadrangular, four-node, Mindlin-type finite-element, with four degrees of freedom per node. Locking phenomena are avoided by adopting a linked interpolation method and enriching the interpolation scheme of the rotational field with some internal degrees of freedom.¹

The middle cross-section $\Omega$ of the plate is discretized by using a regular mesh. The potential energy $E$ and the kinetic energy $T$ of the structure are then evaluated as functions of the nodal values of the unknown fields $w$, $\varphi$ and $v^p$. Finally, a discrete model for the bending vibration of the actuated plate is obtained via the Hamilton principle. The relevant governing equations turn out to be:

$$\begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ \dot{v}_n \end{pmatrix} + \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} \dot{x}_n \\ \dot{v}_n \end{pmatrix} + \begin{pmatrix} K_{mm} & K_{me} \\ K_{me}^T & K_{ee} \end{pmatrix} \begin{pmatrix} x_n \\ v_n \end{pmatrix} = \begin{pmatrix} f_n \\ -q_n \end{pmatrix} \quad (13)$$

Here a superscript $T$ denotes transposition, $x_n$, $v_n$, $f_n$ and $q_n$ are respectively the nodal mechanical degrees of freedom, the nodal electric potentials of the actuator, the nodal external forces, and the nodal electric charges. Moreover, $M$ is the mass matrix, $K_{mm}$ is the stiffness matrix, $-K_{ee}$ is the permittivity matrix and $K_{me}$ is the piezoelectric coupling matrix. Finally, $D$ takes into account the mechanical damping of the structure. In what follows a proportional damping is considered.
All the electroded nodes are then constrained to have the same electric potential. Accordingly, equation (13) is transformed into

\[
\begin{pmatrix}
M & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & L
\end{pmatrix}
\begin{pmatrix}
\ddot{x}_n \\
\ddot{v} \\
\ddot{q}
\end{pmatrix}
+ \begin{pmatrix}
D & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & R
\end{pmatrix}
\begin{pmatrix}
\dot{x}_n \\
\dot{v} \\
\dot{q}
\end{pmatrix}
+ \begin{pmatrix}
K_{mm} & K_{me} & 0 \\
K_{me}^T & K_{me} & 0 \\
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
x_n \\
v \\
q
\end{pmatrix}
= \begin{pmatrix}
f_n \\
-q
\end{pmatrix}
\]

(14)

where \(q\) denotes the electric charge on the actuator, \(v\) is the difference of electric potential between the electrodes, the matrix \(K_{me}\) is transformed into the column vector \(K_{me}\) and the scalar \(C_p\) is the electric capacity of the actuator at fixed structure.

In order to model the influence of the external electric circuit, it is necessary to add to (14) the relevant Kirchhoff equation. Thus, the free vibrations of the coupled electro-mechanical structure are described by the following equations:

\[
\begin{pmatrix}
M & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & L
\end{pmatrix}
\begin{pmatrix}
\ddot{x}_n \\
\ddot{v} \\
\ddot{q}
\end{pmatrix}
+ \begin{pmatrix}
D & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & R
\end{pmatrix}
\begin{pmatrix}
\dot{x}_n \\
\dot{v} \\
\dot{q}
\end{pmatrix}
+ \begin{pmatrix}
K_{mm} & K_{me} & 0 \\
K_{me}^T & K_{me} & 0 \\
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
x_n \\
v \\
q
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

(15)

As it is shown in Figure 1, \(R\) and \(L\) are the values of the passive components of the circuit, and \(v_a\) denotes the applied control voltage.

2.3. Modal analysis

In order to obtain a modal model, the matrices \(M\) and \(K_{mm}\) are simultaneously diagonalized:

\[
V^T M V = I, \quad V^T K_{mm} V = \Lambda
\]

(16)

Here \(V\) is the square matrix whose columns are the eigenmodes of the actuated cantilever plate when the electrodes on the actuator are shorted, \(\Lambda\) is the diagonal matrix containing the squared natural circular frequencies and \(I\) is the identity matrix.

Then, by introducing the modal coordinates \(y = V^{-1} x\), equation (15) is transformed into the following equation:

\[
\begin{pmatrix}
I & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & L
\end{pmatrix}
\begin{pmatrix}
\ddot{y} \\
\ddot{v} \\
\ddot{q}
\end{pmatrix}
+ \begin{pmatrix}
\Delta & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & R
\end{pmatrix}
\begin{pmatrix}
\dot{y} \\
\dot{v} \\
\dot{q}
\end{pmatrix}
+ \begin{pmatrix}
\Lambda & K \\
K^T & -C_p & 1 \\
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
y \\
v \\
q
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
v_a
\end{pmatrix}
\]

(17)

where the column vector \(K\) contains the modal coupling stiffnesses relevant to all the eigenmodes of the structure and \(\Delta\) is a diagonal matrix whose elements are the modal mechanical damping coefficients. Reduced discrete models are obtained from (17) by taking into account only some components of the modal coordinates vector \(y\). In particular, taking into account only the first eigenmode of the structure, the reduced model reads explicitly as:

\[
m \ddot{y}_1 + c_1 \dot{y}_1 + \lambda y_1 + kv = 0
\]

(18)

\[
k y_1 - C_p v = -q
\]

(19)

\[
L \ddot{q} + R \dot{q} + v = v_a
\]

(20)

where \(m = 1, c, \lambda, k\) and \(y_1\) are, respectively, the mass, the mechanical damping coefficient, the stiffness, the coupling stiffness and the modal coordinate relevant to the first eigenmode of the structure. Solving (19) with respect to \(v\) and substituting into (18) and (20), the following two equations are obtained:

\[
m \ddot{y}_1 + c_1 \dot{y}_1 + \lambda y_1 + \frac{k}{C_p} q = 0
\]

(21)

\[
L \ddot{q} + R \dot{q} + \frac{1}{C_p} q + \frac{k}{C_p} y_1 = v_a
\]

(22)

where \(\lambda = \lambda + k^2 / C_p\).
Table 1. Piezoelectric actuator (ACX). Elastic, dielectric and piezoelectric properties

<table>
<thead>
<tr>
<th></th>
<th>$c_{11}$ [GPa]</th>
<th>$c_{12}$ [GPa]</th>
<th>$c_{13}$ [GPa]</th>
<th>$c_{33}$ [GPa]</th>
<th>$c_{44}$ [GPa]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>133.0</td>
<td>77.5</td>
<td>87.0</td>
<td>127.0</td>
<td>26.7</td>
</tr>
<tr>
<td>$\varepsilon_{11}$ [nF/m]</td>
<td>9.97</td>
<td>8.70</td>
<td>-7.22</td>
<td>15.10</td>
<td>13.37</td>
</tr>
<tr>
<td>$\varepsilon_{33}$ [C/m²]</td>
<td>9.97</td>
<td>8.70</td>
<td>-7.22</td>
<td>15.10</td>
<td>13.37</td>
</tr>
<tr>
<td>$e_{31}$ [C/m²]</td>
<td>-7.22</td>
<td>15.10</td>
<td>13.37</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2.4. Controlled structure

An actuated steel cantilever plate ($E = 210$ GPa, $\nu = 0.3$, $\rho = 7850$ Kg/m³), shown in Figure 1, is considered; its sides are 0.250m, 0.04m and 0.0015m. The piezoelectric actuator sides are 0.046m, 0.033m and 0.000127m; its material constants are given in Table 1 and its density is $\rho' = 7700$ Kg/m³.

The actuated cantilever plate is discretized by means of 28x6 elements and its mass and stiffness matrices are evaluated by using the finite-element approach recalled in Section 2.2. Following the procedure illustrated in the previous Section, the modal parameters relevant to the first five eigenmodes of the coupled structure are computed. Their values, corresponding to modal masses of 1 Kg, are reported in Table 2.

The modal mechanical damping coefficients $d_i$ for each eigenmode, are chosen according to the formula $c = 0.002 \sqrt{\lambda}$. The piezoelectric capacity $C_p$ is 126 nF.

3. PASSIVE CONTROL

The vibration passive damping of a structure can be achieved by using an external shunt circuit, whose passive electric components are a resistor $R$ and an inductor $L$ and the active control voltage $v_a$ is set to zero. In practical applications, the problem arises to choose the values of $R$ and $L$ in such a way as to obtain the most effective vibration damping. In order to perform such an optimization, the following dimensionless version of equations (21)-(22) is adopted:

$$\ddot{Y} + 2\nu \ddot{Y} + Y + \kappa \omega Q = 0$$
$$\ddot{Q} + 2\zeta \omega \dot{Q} + \omega^2 Q + \kappa \omega Y = 0$$

where $\omega = \omega_c / \omega_m$, $\omega_c = \sqrt{1/(LC_p)}$, $\omega_m = \sqrt{\lambda/m}$, $\nu = c/(2\sqrt{m\lambda})$, $\zeta = R/(2\omega_L)$ and $\kappa = k/\sqrt{C_p\lambda}$. The modal coupling coefficient $\kappa$ depends only on the material and geometrical characteristics of the coupled vibrating structure. On the other hand, $\omega$ and $\zeta$ depend on the parameters $R$ and $L$ of the external electric circuit.

By adopting a pole-placement technique, the optimal values $\omega_{opt}$ and $\zeta_{opt}$ are implicitly defined by the following equations:

$$\kappa = \sqrt{(1 - \nu^2)(-1 + \omega_{opt}^2 + 2\nu^2 - 2\nu \sqrt{\nu^2 - 1 + \omega_{opt}^2})/\omega_{opt}}$$
$$\zeta_{opt} = \sqrt{\nu^2 - 1 + \omega_{opt}^2}/\omega_{opt}$$

It is easy to verify that in the special case $\nu = 0$, relevant to the situation of vanishing mechanical damping, the above equations reduce to

$$\omega_{opt} = \sqrt{1/(1 - \kappa^2)}$$
$$\zeta_{opt} = \kappa$$

Table 2. Modal parameters of the actuated cantilever plate relevant to the first 5 eigenmodes

<table>
<thead>
<tr>
<th></th>
<th>1st mode</th>
<th>2nd mode</th>
<th>3rd mode</th>
<th>4th mode</th>
<th>5th mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$ [N/m]</td>
<td>1.4587e+04</td>
<td>5.6055e+05</td>
<td>4.4118e+06</td>
<td>1.7264e+07</td>
<td>4.6514e+07</td>
</tr>
<tr>
<td>$k$ [N/V]</td>
<td>-1.0304e+02</td>
<td>2.3515e+02</td>
<td>2.7655e+02</td>
<td>1.8678e+03</td>
<td>-3.8944e+03</td>
</tr>
</tbody>
</table>
which exactly coincide with Hagood and von Flotow’s formulas, up to a transformation between the present dimensionless parameters $\kappa$, $\omega$ and $\zeta$, and Hagood’s $K_{ij}$, $\delta$ and $r$, given by:

\[
\kappa^2 = \frac{K_{ij}^2}{1 + K_{ij}^2}, \quad \omega = \frac{\delta}{\sqrt{1 + K_{ij}^2}}, \quad \zeta = \frac{r\delta}{2}
\]  

(29)

4. ACTIVE CONTROL

The purpose of this Section is to achieve vibration damping for the cantilever plate described in Section 2, by means of a linear dynamic feedback compensator which generates the control input $v_a$ based on past values of the measured output $z = \hat{w}_P$, being $w_P$ the transversal displacement of the location $P$ of the accelerometer. In this Section, and in the following one, we will often refer to the feedback scheme reported in Figure 2, where $S$ represents the plant to be controlled, $z$ the measured output, $K$ the dynamic compensator and $u$ the control input. For the sake of simplicity, the design is based on a reduced-order model of the system, which takes into account only the first eigenmode and neglects the mechanical damping. The design is carried out by means of standard LQG techniques (see, e.g., the books\textsuperscript{13,14}), considering, therefore, a suitable stochastic model of the system. Letting $\eta(t) = [y_1(t) \quad y_1(t)]^T$, be the state of the system at time $t$, the state space equations of our design model are as follows:

\[
\eta(t) = A\eta(t) + b v_a(t) + \xi(t),
\]

(30)

\[
z(t) = C\eta(t) + d v_a(t) + \theta(t),
\]

(31)

where $\xi(t), \theta(t)$ are stationary Gaussian random processes described by $\mathbb{E}\{\xi(t)\} = 0$, $\mathbb{E}\{\theta(t)\} = 0$, $\mathbb{E}\{\xi(t)\xi^T(\tau)\} = \Xi \delta(t - \tau)$ and $\mathbb{E}\{\theta(t)\theta^T(\tau)\} = \Theta \delta(t - \tau)$, being $\Xi \in \mathbb{R}^{2 \times 2}$, $\Xi = \Xi^T$, $\Xi \geq 0$, $\Theta \in \mathbb{R}$, $\Theta > 0$, and with $\mathbb{E}\{\cdot\}$ denoting the expected value of the argument. From equations (21) and (22), setting $R = 0$, $L = 0$, and $d = 0$, we have

\[
A = \begin{bmatrix} 0 & 1 \\ -\lambda & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ -k \end{bmatrix},
\]

where $C = [-\lambda \tilde{V} \quad 0]$, $d = -k \tilde{V}$, being $\tilde{V}$ the transversal displacement of the first eigenmode of the plate in correspondence to the location $P$ of the accelerometer. The LQG controller proposed here is designed in order to minimize the following performance index:

\[
\mathcal{J} = \mathbb{E}\left\{\frac{1}{2} \int_0^{+\infty} (\eta^T(\tau)Q\eta(\tau) + r v_a^2(\tau)) \, d\tau \right\},
\]

where $r \in \mathbb{R}$, $r > 0$ and $Q \in \mathbb{R}^{2 \times 2}$, $Q = Q^T$, $Q \geq 0$, are to be seen, together with $\Theta$ and $\Xi$, as the design parameters, to be chosen in order to obtain a satisfactory behaviour for the closed loop system. As is well known, if $P$ and $\Sigma$ denote the unique positive definite solutions of the algebraic Riccati equations:

\[
Q + A^T P + P A - \frac{1}{r} P b b^T P = 0,
\]

\[
\Xi + A \Sigma + \Sigma A^T - \frac{1}{\Theta} \Sigma C^T C \Sigma = 0,
\]

respectively, the LQG compensator is described, in state-space form, by the equations:

\[
\dot{x}_K(t) = A_K x_K(t) + b_K z(t),
\]

(32)

\[
u_a(t) = C_K x_K(t),
\]

(33)
where \( x_K \in \mathbb{R}^2 \), \( A_K = A + bK_c - K_o C - K_o dK_c \), \( b_K = K_o \), \( C_K = K_c \), \( K_c = \frac{1}{r} b^T P \), \( K_o = \frac{1}{\Theta} \Sigma C^T \) and \( u_a \) would coincide with \( v_a \) if it was possible to directly connect the proposed compensator with the given system. The choice of \( r, Q, \Theta \) and \( \Xi \) has to be made by considering the well known trade-off between the objective of increasing the convergence speed, which requires that the elements of \( Q \) and \( \Xi \) are taken "large", compared to \( r \) and \( \Theta \), respectively, and the objectives of keeping the absolute value of \( v \) suitable small (i.e., admissible for the piezoelectric actuator) and of reducing the "spillover" effects due to the unmodelled dynamics (which will arise when the compensator designed on the basis of the reduced order model (30)-(31) will be applied to the actual infinite dimensional system). This two last objectives, in turn, require that the elements of \( Q \) and \( \Xi \) are "small", compared to \( r \) and \( \Theta \), respectively. Without loss of generality, we have set \( r = \Theta = 1 \), whereas, in order to select matrices \( Q \) and \( \Xi \), we have found very convenient to use the technique reported by Kraus and Kučera,\(^7\) which allows to select such matrices in order to obtain desired closed-loop eigenvalues for the ideal overall system, obtained from the block diagram reported in Figure 2 by replacing \( S \) with the model given by (30) and (31) and \( K \) with the compensator given by (32) and (33). Satisfactory results have been obtained with

\[
Q = \begin{bmatrix} 41.3 & 0 \\ 0 & 0.0279 \end{bmatrix} \quad \text{and} \quad \Xi = \begin{bmatrix} 0.001 & 0 \\ 0 & 6.15 \end{bmatrix} \times 10^{-13},
\]

yielding, for the mentioned ideal feedback connection, the closed loop eigenvalues \( \lambda_{1,2} = -10 \pm 122j \) and \( \lambda_{3,4} = -15 \pm 122j \). In order to evaluate the performance of the proposed LQG compensator when applied to the real system, simulations have been performed by using a reduced order model which takes into account the first five bending vibration modes of the plate, and includes some mechanical damping. Furthermore, to be more realistic, it has been taken into account that, in order to preserve the piezoelectric actuator, it must be ensured that \( |v| \leq 100 V \). In an experimental setup such a saturation is guaranteed by the power amplifier which generates the input voltage, but in the simulations an ideal saturation has been considered:

\[
u_a(t) = \begin{cases} 
100, & \text{if } u_a(t) > 100, \\
u_a(t), & \text{if } -100 \leq u_a(t) \leq 100, \\
-100, & \text{if } u_a(t) < -100,
\end{cases}
\]

(34)

being \( u_a(t) \) the output of the linear LQG compensator at time \( t \). Therefore, by referring again to the block diagram in Figure 2, the closed-loop system that is simulated hereafter can be obtained by replacing \( S \) with the cascade connection of a nonlinear, static subsystem representing the saturation, and the reduced order model of the plate taking into account five bending vibration modes and the mechanical damping, and \( K \) with the LQG compensator given by (32) and (33). The results of the first two seconds of simulation, starting from null initial conditions for the compensator, and initial deformations of the plate corresponding to a suitable impulsive force applied at time \( t = 0 \) to the free end of the plate in central position, have been used to compute the values of damping of the first five bending vibration modes of the plate, reported in the second line of Table 3, labelled \( ACT \) (Active-Continuous-Time control). In order to be implemented through a digital computer, the proposed continuous-time LQG compensator needs to be discretized. In particular, assuming that the sampling time is \( \delta_T = 1/1024 \text{s} \), which is an admissible value for an experimental equipment, we have performed the discretization by means of the MATLAB routine \( c2d \), which gives the matrices \( A_D, b_D, \) and \( C_D \) of the discrete time compensator described by:

\[
zd(K+1) = A_D zd(k) + b_D zd(k),
\]

(35)

\[
u_a,D(k) = C_D zd(k),
\]

(36)

as functions of the matrices \( A_K, b_K \) and \( C_K \), and of the sampling time \( \delta_T \). Notice that the discrete-time input \( zd(k) \) of the discretized LQG compensator is the ideal sampling of the continuous-time output of the plant \( z(t) \), i.e., \( zd(k) = z(k \delta_T) \), whereas a zero order holding mechanism converts the discrete-time output \( u_a,D(k) \) of the discretized compensator into the continuous-time control input to be saturated before being applied to the plant, i.e., \( u_a(t) = u_a,D(k) \), for all \( t \in (k \delta_T, (k + 1) \delta_T] \), and the actual input \( v_a \) can be computed according to (34). Therefore, by referring again to the block diagram in Figure 2, the closed-loop system that is simulated hereafter can be obtained by replacing \( S \) with the cascade connection of a zero-order holder, a nonlinear, static subsystem representing the saturation, the reduced order model of the plate taking into account five bending vibration modes and the mechanical damping and an ideal sampler, and \( K \) with the discrete-time compensator given by (35) and (36). The results of the first two seconds of simulation, starting from null initial conditions for the compensator,
Figure 3. Simulation results for the Active Discrete-Time control law. On the left: input voltage \(v_a\). On the right: transversal displacement of the cantilever tip.

Figure 4. Comparison between the results of Passive control (grey line) and the ones of Active Discrete-Time control (black line). The first two seconds of simulation are reported on the left, for the sake of completeness. In the detail on the right it can be appreciated that, once the effects of the saturation disappear, active control provides better vibration damping.

and from the same initial conditions of the plate used in the ACT simulation, are reported in Figure 3. In order to compare such simulation results with the ones that can be obtained by means of purely passive control laws, we have simulated, on the same five-degrees of freedom reduced order model, the behaviour of the passive control law proposed in Section 3, from the same initial conditions at time \(0^+\). The comparison between the behaviour of the discretized LQG compensator and the passive control law discussed in Section 3 is reported in Figure 4.

5. SEMI-ACTIVE CONTROL

In this Section, we study the performance of a semi-active control of the cantilever plate described in Section 2, i.e., we assume that an electrical circuit like the one described in Section 3 is connected in series between one of the electrodes of the piezoelectric actuator and the power amplifier which provides the active input voltage \(v_a\) (see Figure 1). Therefore, the voltage \(v\) actually applied to the piezoelectric actuator is given by \(v = -(L\dot{q} + Rq) + v_a\). Our purpose is to design a linear dynamic feedback compensator which generates the control input \(v_a\) based on past values of the measured output \(z = \ddot{w}_P\), in order to achieve, if possible, better vibration damping than the one
obtained by means of either purely passive or purely active control, without rendering unduly complicate the design process. The central idea of this Section is that of tuning the electrical parameters \( R \) and \( L \) in order to optimally damp the second vibration mode of the vibrating structure, so that the design of the compensator can be based again on a reduced order model of the cantilever plate which only takes into account the first vibration mode, being the second one taken care by the electrical circuit. However, the reduced order model to be used for the design must take into account the presence of the electrical components, so that, letting \( \eta(t) = [y_1(t) \ y_1(t) \ q(t) \ \dot{q}(t)] \), be the state of the system at time \( t \), the state space equations of our design model are as follows:

\[
\begin{align*}
\dot{\eta}_e(t) &= A_e \eta_e(t) + b_e v_a(t) + \xi_e(t), \\
z(t) &= C_e \eta_e(t) + d_e v_a(t) + \theta(t),
\end{align*}
\]

(37)  (38)

where \( \xi_e(t) \) is a stationary Gaussian random process described by \( \mathbb{E}\{\xi_e(t)\} = 0 \) and \( \mathbb{E}\{\xi_e(t)\xi_e^T(\tau)\} = \Xi_e \delta(t-\tau) \), being \( \Xi_e \in \mathbb{R}^{4 \times 4} \), \( \Xi_e = \Xi_e^T \), \( \Xi_e \geq 0 \), and \( \theta \) is as in Section 4. From equations (21) and (22), we have

\[
A_e = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-\lambda - k^2/C^p & 0 & -k/C^p & 0 \\
0 & 0 & 0 & 1 \\
-k/(LC^p) & 0 & -1/(LC^p) & -R/L
\end{bmatrix}, \\
b_e = \begin{bmatrix}
0 \\
0 \\
0 \\
1/L
\end{bmatrix}, \\
C_e = \tilde{V} \begin{bmatrix}
-\lambda - k^2/C^p & 0 & -k/C^p & 0
\end{bmatrix}, \\
d_e = 0.
\]

The LQG controller proposed here is designed in order to minimize the following performance index:

\[
J_e = \mathbb{E}\left\{ \frac{1}{2} \int_{0}^{+\infty} (\eta_e^T(\tau)Q_e \eta_e(\tau) + rv_a^2(\tau)) \, d\tau \right\},
\]

where \( Q_e \in \mathbb{R}^{4 \times 4} \), \( Q_e = Q_e^T \), \( Q_e \geq 0 \), can be chosen, together with \( \Xi_e \), in order to obtain a satisfactory transient behaviour for the closed-loop system, without forgetting the constraints deriving from the need for maintaining low enough the control effort, and that of minimizing spillover effects (as mentioned, \( r \) can be chosen equal to 1, without loss of generality). The choice we perform here, in order to emphasize the nice behaviour of the semi-active control scheme, is that of using as matrices \( Q_e \) and \( \Xi_e \) the more natural extensions of the matrices \( Q \) and \( \Xi \) used in Section 4:

\[
Q_e = \begin{bmatrix}
41.3 & 0 & 0 & 0 \\
0 & 0.0279 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad \text{and} \quad \Xi_e = \begin{bmatrix}
0.001 & 0 & 0 & 0 \\
0 & 6.150 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \times 10^{-13}.
\]

In order to evaluate the performance of the proposed semi-active control law when applied to the real system, simulations have been performed by using the same reduced order model used for the simulations in Section 4, obviously extended in order to take into account the electrical circuit, with \( R = 947 \Omega \) and \( L = 13.6 \text{H} \). As in Section 4 the output voltage computed by the LQG compensator has been saturated according to equations (34).

Notice that, in this case, the voltage actually applied to the piezoelectric element is not simply \( v_a \) but rather \( v_a - (L\ddot{q} + R\dot{q}) \). Therefore, by referring again to the block diagram in Figure 2, the closed-loop system that is simulated hereafter can be obtained by replacing \( S \) with the cascade connection of a nonlinear, static subsystem representing the saturation, the electrical circuit and the reduced order model of the plate taking into account five bending vibration modes and the mechanical damping, and \( K \) with the fourth-order LQG compensator designed above. The results of the first two seconds of simulation, starting from null initial conditions for the compensator and the electrical circuit, and the same initial conditions for the plate already used in the previous simulations, have been used to compute the values of damping of the first five bending vibration modes of the plate, reported in the fourth line of Table 3, labelled SACT (Semi-Active Continuous-Time control).

In order to be implemented through a digital computer, the proposed continuous-time LQG compensator has been discretized, following exactly the same procedure used in Section 4, thus obtaining a discrete-time fourth order compensator. Therefore, by referring again to the block diagram in Figure 2, the closed-loop system that is simulated hereafter can be obtained by replacing \( S \) with the cascade connection of a zero-order holder, a nonlinear, static subsystem representing the saturation, the electrical circuit, the reduced order model of the plate taking into account five bending vibration modes and the mechanical damping and an ideal sampler, and \( K \) with the discrete-time fourth order compensator obtained by discretization. The results of the first two seconds of simulation, starting
Figure 5. Simulation results for the Semi-Active Discrete-Time control law. On the left: input voltage $v_a$. On the right: transversal displacement of the cantilever tip.

Figure 6. Comparison between the results of Passive control (grey lines) and Semi-Active Continuous-Time control (on the left) and Semi-Active Discrete-Time control (on the right).

from null initial conditions for the compensator, and from the same initial conditions of the plate used in the previous simulations, are reported in Figure 5.

In order to compare such simulation results with the ones that can be obtained by means of purely passive control laws, we have simulated again, on the same five-degrees of freedom reduced order model, the behaviour of the passive control law proposed in Section 3, from the same initial conditions at time $0^+$. The results of such a simulation are reported, together with the ones relative to the Semi-Active control law, both in its continuous-time and its discrete-time version, in Figure 6. Finally, in order to evaluate numerically the capabilities of damping the first vibration mode of the plate, and the related "spillover" effects, we have computed (by means of Fast Fourier Transform) the power spectral density of the tip position $w_1$ corresponding to the open-loop system and to the five simulations described above, related to different control laws. We have evaluated the attenuation (in decibel) of the first five bending vibration modes of the plate, reporting the results in Table 3. In order to reduce the effects of saturation in the comparison, when computing the FFT of such signals we have considered a time interval, included in the first two seconds of simulation, starting after time $0.3s$. 

310
<table>
<thead>
<tr>
<th>Control law</th>
<th>1st mode</th>
<th>2nd mode</th>
<th>3rd mode</th>
<th>4th mode</th>
<th>5th mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>11.1258</td>
<td>-0.2408</td>
<td>-0.0018</td>
<td>-0.4128</td>
<td>-1.7312</td>
</tr>
<tr>
<td>ACT</td>
<td>30.5266</td>
<td>1.8145</td>
<td>-1.2864</td>
<td>-3.0295</td>
<td>8.9642</td>
</tr>
<tr>
<td>ADT</td>
<td>17.1488</td>
<td>2.7116</td>
<td>0.3946</td>
<td>-0.3024</td>
<td>20.8325</td>
</tr>
<tr>
<td>SACT</td>
<td>35.6206</td>
<td>53.2315</td>
<td>-0.0208</td>
<td>-0.4323</td>
<td>-1.6807</td>
</tr>
<tr>
<td>SADT</td>
<td>17.2378</td>
<td>6.5124</td>
<td>-0.0669</td>
<td>-0.3530</td>
<td>-1.9991</td>
</tr>
</tbody>
</table>

Table 3. Modal attenuations in decibel obtained by Passive (P), Active Continuous Time (ACT), Active Discrete Time (ADT), Semi-Active Continuous Time (SACT) and Semi-Active Discrete Time (SADT) control laws. Negative values correspond to “spillover” effects.

6. CONCLUSIONS

Vibration damping of a cantilever plate by means of a piezoelectric actuator was studied. The vibrating electromechanical coupled structure was modeled by using a suitable finite-element formulation together with a modal reduction. A purely passive single-mode control composed of an optimally tuned external RL shunt circuit and a purely active control based on classical LQG techniques were considered. Then, a semi-active control was designed, obtained by tuning the external shunt circuit on the second vibration mode of the structure and using a LQG controller designed on a reduced-order model. The latter supplied, in simulation, very satisfactory results with respect to the purely passive or purely active control strategies.

ACKNOWLEDGMENTS

The authors wish to thank professor F. Maceri for valuable comments on this paper. The financial supports of CNR and MURST are gratefully acknowledged.

REFERENCES